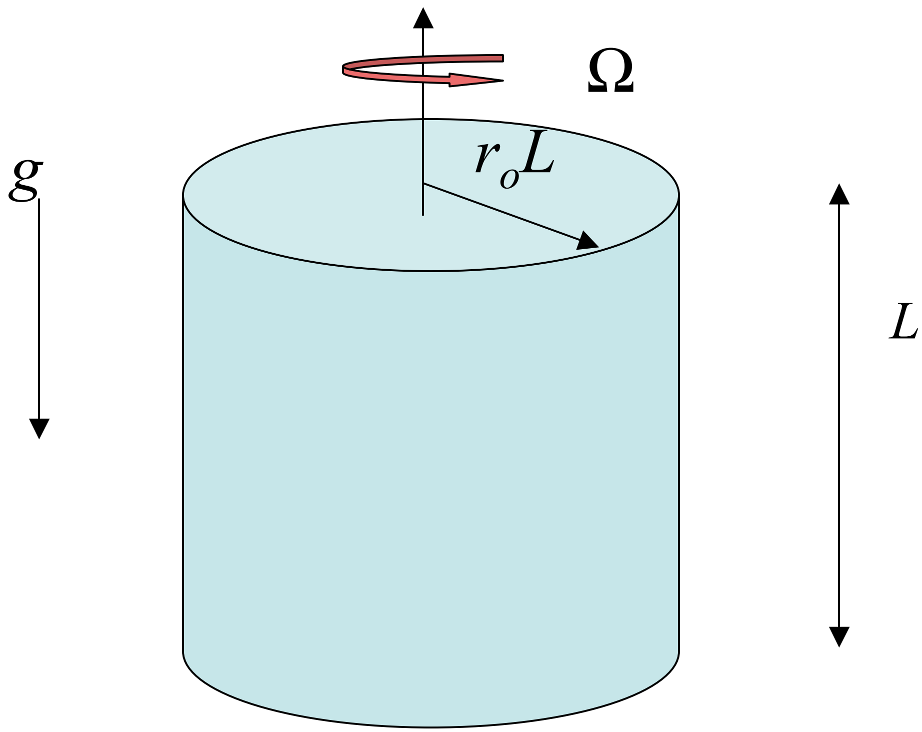
GFD 2007
Boundary Layers:Stratified Fluids

- The presence of density stratification introduces new and very interesting elements to the boundary layer picture and the nature of the control of the interior flow by the boundary layers. As we discussed previously, the vertical velocity pumped out of (or into) the Ekman layers on horizontal surfaces effected a control on the evolution of the interior, for example , leading to spin-down.

- **One can think of that control as proceeding along the vortex lines of the overall rotation of the fluid as these are the lines along which information is propagated (by inertial waves)** for a homogeneous , uniformly rotating fluid.

- When stratification is added information also propagates laterally, by internal gravity waves (assuming the stratification is gravitationally stable) and frequently this direction is perpendicular to the rotation direction **so one can anticipate a kind of competition between rotation and stratification in determining the nature of the boundary layer influence** on the bulk of the fluid. This will be the emphasis of our present discussion.

The cylinder problem.
(tip of the hat to the MJQ)



Fluid heated to establish a stable stratification when there is no relative motion in the cylinder.

$$T = \Delta T_v z / L$$

Assumes that rotation is small and doesn't bend the isotherms, i.e.

$$\Omega^2 L / g << 1$$

Scaling

$$\vec{u}_* = U\vec{u}, \;\; \vec{x}_* = L\vec{x}, \quad T_* = \Delta T_v (z_* / L) + \Delta T_h T(x,y,z)$$

$$\rho_* = \rho_o \left[1 - \alpha\left(\Delta T_v z + \Delta T_h T\right)\right]$$

$$p_* = \rho_o g \alpha \Delta T_v z_*^{\,2} / 2L + \rho_o f U L \;\; p(x,y,z)$$

$f=2\Omega$

choose scaling velocity

$$U = \frac{\alpha g \Delta T_h}{f} \qquad \text{(thermal wind)}$$

Motion is incompressible and Boussinesq

Equations of motion (steady)

$$\varepsilon \vec{u} \cdot \nabla \vec{u} + \hat{k} \times \vec{u} = -\nabla p + T\hat{k} + \frac{E}{2} \nabla^2 \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

$$\varepsilon \vec{u} \cdot \nabla T + wS = \frac{E}{2\sigma} \nabla^2 T$$

$$\boxed{\varepsilon = \frac{U}{fL}, \quad E = \frac{2\nu}{fL^2}, \quad \sigma = \frac{\nu}{\kappa}, \quad S = \frac{\alpha g \Delta T_v}{f^2 L} \equiv \frac{N^2}{f^2}}$$

$$\frac{\varepsilon}{S} = \frac{\Delta T_h}{\Delta T_v}$$

We will generally assume this ratio is small.

Boundary conditions

No slip, no normal flow

One or more boundaries may be moving.

Cylinder walls either insulating or at a fixed, given temperature.

The linear problem

We will consider a problem in which the fluid is driven so gently that the Rossby number is small enough to allow the neglect of all nonlinear terms.

When the flow occurs in a cylinder and the forcing and the motion are axially symmetric the neglect of the nonlinearity is generally sensible. It is usually the advent of instabilities of the flow we are going to describe that more seriously limit the validity of the linearization

Coordinate system r, θ, z with corresponding velocities u, v, w

Linear equations of motion in cylindrical coordinates. Axially symmetric motions

$$-v = -p_r + \frac{E}{2}\left[\nabla^2 u - u / r^2\right],$$

$$u = \qquad\qquad \frac{E}{2}\left[\nabla^2 v - v / r^2\right],$$

$$0 = -p_z + T + \frac{E}{2}\nabla^2 w,$$

$$\frac{1}{r}(ru)_r + w_z = 0,$$

$$w\sigma S = \frac{E}{2}\nabla^2 T,$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The Ekman layers

Again $\zeta = z/E^{1/2}$, $w = E^{1/2}W$ near z=0 the Ekman boundary layer would satisfy

$$-v = -p_r + \frac{1}{2}u_{\zeta\zeta},$$

$$u = \frac{1}{2}v_{\zeta\zeta},$$

$$0 = -p_\zeta + E^{1/2}T + \frac{1}{2}E\ W_{\zeta\zeta},$$

$$\sigma S E^{1/2} W = \frac{1}{2}T_{\zeta\zeta}$$

$$(ru)_r + rW_\zeta = 0.$$

if

$$\sigma S \le E^{-1/2}$$

We can ignore buoyancy forces in the Ekman layer. They are so thin they remain unchanged.

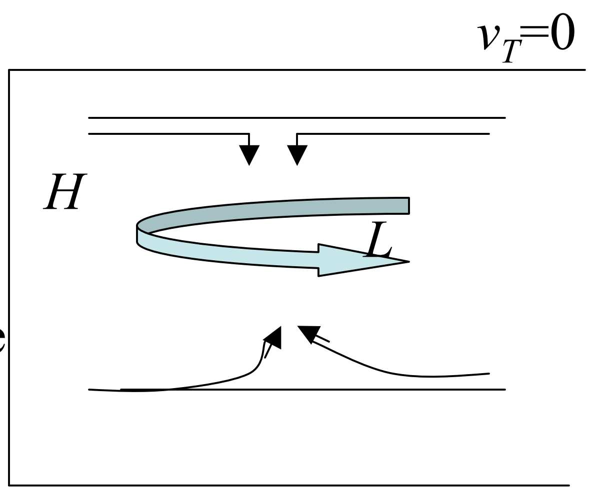
True only if boundary is horizontal!!

The compatibility condition

At lower boundary

$$w_I(r,0) = E_v^{1/2} W(r,\infty) = \frac{E_v^{1/2}}{2r}(rv_I)_r$$

Let the upper boundary, or lid, of the cylinder be rotating with the differential speed $v_T(r)$. Then the same analysis at the upper boundary yields



$$w_I(r,1) = \frac{E_v^{1/2}}{2}\frac{1}{r}\frac{\partial}{\partial r} r\left(v_T - v_I(r,1)\right)$$

At upper boundary, z =1

Interior equations $E << 1$

$$v_I = p_{I r}, \qquad T_I = -p_{I_z},$$

$$u_I = \frac{E}{2}(\nabla^2 v_I - v_I / r^2)$$

$$(r u_I)_r + w_{I_z} = 0$$

$$w_I = \frac{E}{2\sigma S}\nabla^2 T_I$$

Note that u_I is $O(E)$

If $\sigma S < 1$ then w_I is much larger than u_I so that

$$\frac{\partial w_I}{\partial z} = 0$$

Thus w_I is equal to its average at $z = 0$ and $z = 1$, i.e.

$$w_I = \frac{1}{2}\left[w_I(r,1) + w_I(r,0)\right]$$

$$= \frac{E^{1/2}}{4}\frac{1}{r}\frac{\partial}{\partial r} r\left[v_T - \{v_I(r,1) - v_I(r,0)\}\right]$$

The vertical velocity and the thermal wind relation

$$\frac{\partial v_I}{\partial z} = \frac{\partial T_I}{\partial r},$$ Thermal wind

Leading to:

$$v_I(r,1) - v(r,0) = \frac{\partial}{\partial r}\int_0^1 T_I(r,z')dz'$$

So the vertical velocity becomes given directly in terms of the temperature and external forcing

$$\boxed{w_I = -\frac{E^{1/2}}{4}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\int_0^1 T_I(r,z')\right) + \frac{E^{1/2}}{4}\frac{1}{r}\frac{\partial}{\partial r}\left(rv_T\right)}$$

$$\sigma S << 1$$

Interior thermal equation

$$\frac{E}{2\sigma S}\nabla^2 T_I + \frac{E^{1/2}}{4}\nabla^2\int_0^1 T_I(r,z')dz' = \frac{E^{1/2}}{4}\frac{1}{r}\frac{\partial}{\partial r}\left(rv_T\right)$$

we define the pseudo temperature θ

$$\theta = T_I + \lambda\int_0^1 T_I dz'.$$

$$\boxed{\lambda = \frac{\sigma S}{2E^{1/2}}}$$

$$\boxed{\nabla^2\theta = \lambda\frac{1}{r}\frac{\partial}{\partial r}\left(rv_T\right)}$$

The temperature in terms of the pseudo temperature

$$T_I = \theta - \frac{\lambda}{1+\lambda}\int_0^1 \theta dz$$

$$\nabla^2\theta = \lambda \frac{1}{r}\frac{\partial}{\partial r}\left(r v_T\right)$$

We need boundary conditions on all the side walls of the cylinder to determine the solution.

For $\sigma S < 1$ the streamfunction

the temperature deviation in the Ekman layer is O($E^{1/2}\sigma S$), and, as we shall see, there is no other boundary layer on the horizontal boundaries for σS <<1 so the interior temperature must satisfy the boundary conditions on the horizontal boundaries. However, we need to consider the boundary layers on the side wall to find the appropriate boundary conditions for the pseudo temperature equation on $r=r_0$.

Useful to introduce streamfunction for u and w

$$ru = -\psi_z,$$

$$rw = \psi_r$$

For the interior, from

$$\boxed{w_I = -\frac{E^{1/2}}{4}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\int_0^1 T_I(r,z')\right) + \frac{E^{1/2}}{4}\frac{1}{r}\frac{\partial}{\partial r}(rv_T)}$$

$$r\psi_I(r) = \frac{E^{1/2}}{4}rv_T - \frac{E^{1/2}}{4(1+\lambda)}r\frac{\partial}{\partial r}\int_0^1 \theta dz$$

The side wall boundary layers $\boldsymbol{\sigma S \ll 1}$

Write all variables as their interior values plus a boundary layer *correction* that by definition vanishes for large values of the boundary layer coordinate e.g.

$$u = u_I(r) + u_{bl}(\xi), \qquad u_{bl} \to 0, \;\; \xi \to \infty$$

$$v = p_r$$

$$u = \frac{E}{2} v_{rr}$$

$$0 = -p_z + T + \frac{E}{2} w_{rr}$$

$$u_r + w_z = 0$$

$$\sigma S w = \frac{E}{2} T_{rr}$$

Boundary layer balances for the correction functions. The azimuthal velocity v remains in geostrophic balance.

References

Barcilon,V. and J. Pedlosky 1967 a, Linear theory of rotating stratified fluid motions. *J. Fluid. Mech.*,**29**, 1-16.

____________________________b, Unified linear theory of homogeneous and stratified rotating fluids. *J. Fluid Mech.,* ***29****,609-621.*

Pedlosky,J., J.A. Whitehead, and G. Veitch, 1997. Thermally driven motions in a rotating stratified fluid. Theory and experiment. *J. Fluid Mech.,* **339**,391-411.

Veronis, G. Analogous behavior of rotating and stratified fluids. *Tellus*, **19**, 620-634

Whitehead, J.A. and J. Pedlosky 2000, Circulation and boundary layers in differentially heated rotating stratified fluid. *Dyn. Atmospheres and Oceans*, **31**, 1-21

The Master Equation

Eliminate all variables with respect to the pressure to obtain:

$$\overbrace{E^2 p_{6r} / 4}^{a} + \overbrace{\sigma S p_{rr}}^{b} + \overbrace{p_{zz}}^{c} = 0$$

We assume that all z derivatives are O(1) and r derivatives give us a term δ^{-1}

So term (a) is O(E^2 / δ^6)

To satisfy all the boundary conditions the full order of the equations must be maintained.

The Stewartson Layer

Now we can ask what balances are possible and we note that the full order of the equation (6th order in r) must be preserved in any parameter setting. Let's suppose that terms (a) and (c) are of the same order. This clearly implies that

$$\overbrace{E^2 p_{6r} / 4}^{a} + \overbrace{\sigma S p_{rr}}^{b} + \overbrace{p_{zz}}^{c} = 0$$

$$E^2 / \delta^6 = O(1), \qquad \Rightarrow \delta = E^{1/3}$$

This layer exists for a homogeneous fluid when S is zero. To find the parameter limit of validity of that balance, we can evaluate the order of term (b) and compare it to terms (a) and (c). Term (b) is

$$\sigma S / E^{2/3}$$

So for the balance a-c to hold we need

$$\sigma S < E^{2/3}$$

The buoyancy layer(1)

If the inequality is reversed the a-b balance is no longer valid.

$$\overbrace{E^2 p_{6r}/4}^{a} + \overbrace{\sigma S p_{rr}}^{b} + \overbrace{p_{zz}}^{c} = 0$$

$$\delta = \frac{E^{1/2}}{(\sigma S)^{1/4}}$$

From the a-b balance

This is the **buoyancy layer** thickness

$$\delta_* = L\delta = \frac{(\nu\kappa)^{1/4}}{N^{1/2}}$$

Dimensional: independent of rotation

The ratio of term c to term b is

$$\frac{1}{\sigma S/\delta^2} = \frac{E}{(\sigma S)^{3/2}}$$

And so the buoyancy layer balance is valid when $\sigma S > E^{2/3}$

The Hydrostatic layer (1)

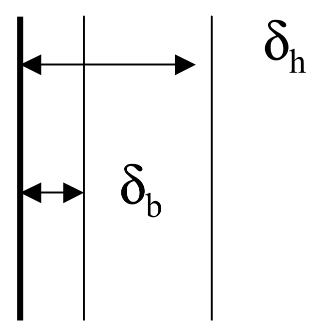
The balance b-c yields $\delta = (\sigma S)^{1/2}$ $\qquad$ $\overbrace{E^2 p_{6r}/4}^{a} + \overbrace{\sigma S p_{rr}}^{b} + \overbrace{p_{zz}}^{c} = 0$

And this too requires $\sigma S > E^{2/3}$ $\qquad$ and, of course, $\sigma S < 1$

When $\sigma S = E^{2/3}$ the hydrostatic layer and the buoyancy layer merge to become the $E^{1/3}$ layer

or , as the stratification increases the Stewartson layer splits into two sublayers, the hydrostatic layer and the buoyancy layer. Note that the ratio of their two thicknesses is:

$$\frac{\delta_{buoyancy}}{\delta_{hydrostatic}} = \frac{E^{1/2}}{(\sigma S)^{1/4}(\sigma S)^{1/2}} = \left(\frac{E^{2/3}}{\sigma S}\right)^{3/4} << 1$$



The overall balances

$$\sigma S > E^{2/3}$$

Buoyancy layer

Hydrostatic layer

$$\overbrace{E^2 p_{6r}/4}^{a} + \overbrace{\sigma S p_{rr}}^{b} + \overbrace{p_{zz}}^{c} = 0$$

Stewartson layer

$$\sigma S < E^{2/3}$$

The overall sixth order system is preserved even when the boundary layer splits in two

The hydrostatic layer balances (1)

$$\delta_h L = \left(\frac{\nu}{\kappa}\right)^{1/2} \frac{N}{f} L$$

In dimensional units. Aside from Prandtl number, this is the Rossby deformation radius.

$$\eta = \frac{r_o - r}{(\sigma S)^{1/2}}$$

Boundary layer coordinate

Correction functions in hydrostatic layer

$v_h = \tilde{v}(\eta, z)$ implies $u_h = \dfrac{E}{\sigma S}\tilde{u}(\eta, z)$ Momentum equation.

$p_h = (\sigma S)^{1/2}\, \tilde{p}$ From geostrophy

It is easy to show that the frictional term in the vertical equation is negligible, hence the layer is in hydrostatic balance.

$$T_h = (\sigma S)^{1/2}\, \tilde{T}$$

Hydrostatic layer balances (2)

From continuity equation

$$w_h = \frac{E}{(\sigma S)^{3/2}} \tilde{w} \qquad \psi_h = \frac{E}{\sigma S} \tilde{\psi}(\eta, z)$$

Ignoring terms of order $E^{2/3} \big/ \sigma S$ or smaller the governing equations in this layer are:

$$\tilde{v} = -\tilde{p}_\eta, \quad \tilde{u} = \frac{1}{2} \tilde{v}_{\eta\eta}$$

$$\tilde{T} = \tilde{p}_z, \qquad \tilde{w} = \frac{1}{2} \tilde{T}_{\eta\eta} \longrightarrow -r_o \tilde{w} = \tilde{\psi}_\eta = -\frac{r_o}{2} \tilde{T}_{\eta\eta}$$

$$\tilde{u}_\eta = \tilde{w}_z$$

so

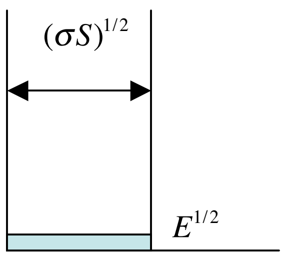
$$\tilde{\psi} = -\frac{r_o}{2} \tilde{T}_\eta$$

$$\boxed{\tilde{v}_{\eta\eta} + \tilde{v}_{zz} = 0}$$

bdy layer eqn. is a pde.

The hydrostatic layer eigenvalue problem (1)

The width of the hydrostatic layer is much greater than the Ekman layer thickness as long as $\sigma S >> E$



So the Ekman layer will act as if the hydrostatic layer is part of the interior and the ordinary compatibility condition will apply,

$$w_h = \frac{E}{(\sigma S)^{3/2}}\tilde{w} = \frac{E}{2(\sigma S)^{3/2}}\tilde{T}_{\eta\eta} = -\frac{E}{2(\sigma S)^{3/2}}\tilde{v}_{z\eta}$$

heat eqn. + thermal wind

$$= \frac{E^{1/2}}{2}v_{h_r} = -\frac{E^{1/2}}{2(\sigma S)^{1/2}}\tilde{v}_\eta$$

Ekman pumping related to vorticity

$$\tilde{v}_z = \pm 2\lambda\tilde{v} \qquad \text{At } z = \binom{0}{1}$$

Hydrostatic layer eigenvalue problem (2)

letting

$$\tilde{v} = V(z)e^{-a\eta}$$

leads to an eigenvalue problem for $\{a, V(z)\}$

$$V_{zz} + a^2 V = 0$$

$$V_z = \mp 2\lambda V \quad at \quad z = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad \lambda = \frac{\sigma S}{2E^{1/2}}$$

Note that if the stratification is large v=0 at the boundary and if the stratification is weak it is the shear that vanishes.

If λ is very small it is easy to show that
the lowest eigenvalue is $a_o \approx \left(\dfrac{2\sigma S}{E^{1/2}}\right)^{1/2}$

and this gives rise to a solution nearly independent of z with a characteristic scale in the radial direction of $E^{1/4}$. This calculation is left to the student. This is the second boundary layer scale of Stewartson for homogeneous fluids

The buoyancy layer (1)

$$\delta_b = \frac{E^{1/2}}{(\sigma S)^{1/4}}$$

Given the scale we can deduce the relative balances in this thin sub layer

$$T_b = \delta_b \hat{T}, \quad w_b = \frac{E}{\sigma S \delta_b} \hat{w}, \; u_b = \frac{E}{\sigma S} \hat{u},$$

$$v_b = \frac{E}{(\sigma S)^{3/2}} \hat{v}, \qquad p_b = \frac{E}{\delta_b (\sigma S)^{3/2}} \hat{p}$$

Boundary layer coordinate:

$$\xi = {(r_o - r)}/{\delta_b}$$

$$\hat{v} = -\hat{p}_\xi, \;\; \hat{u} = \frac{1}{2} \hat{v}_{\xi\xi}, \;\; \hat{w} = \frac{1}{2} \hat{T}_{\xi\xi}, \;\; \hat{u}_\xi = \hat{w}_z,$$

$$0 = \hat{T} + \frac{1}{2} \hat{w}_{\xi\xi}$$

The vertical equation is a balance between buoyancy and friction:

The layer is **non-hydrostatic**

The buoyancy layer (2)

$$\psi_b = \frac{E}{\sigma S}\hat{\psi}, \qquad \hat{\psi} = -\frac{r_o}{2}\hat{T}_\xi$$

$$\hat{T}_{4\xi} + 4\hat{T} = 0$$

ode and like Ekman layer

$$\hat{T} = Ae^{-\xi}\cos\xi + Be^{-\xi}\sin\xi,$$

$$\hat{w} = Ae^{-\xi}\sin\xi - Be^{-\xi}\cos\xi$$

yields velocity normal to boundary and depends on variation vertically along the side wall

$$\hat{u} = -A_z e^{-\xi}(\cos\xi + \sin\xi)/2 + B_z e^{-\xi}(\cos\xi - \sin\xi)/2$$

equivalently

$$\hat{\psi} = \frac{-r_o}{2}\left[-Ae^{-\xi}\{\cos\xi + \sin\xi\} + Be^{-\xi}\{\cos\xi - \sin\xi\}\right]$$

Matching at $r = r_o$: an example

Our goal is to find how the boundary layers translate the physical conditions on the boundary to conditions on the interior problem, i.e. how do the boundary layers control the interior (and vice-versa)

At the rim,using the sum of the interior and boundary layer corrections:

last term negligible

$$v_I(r_o,z) + \tilde{v}(0,z) + \frac{E}{(\sigma S)^{3/2}}\hat{v}(0,z) = 0$$ no slip on v

$$T_{I_r} - \tilde{T}_\eta - \hat{T}_\xi = 0,$$ side wall **insulated**

$$\psi_I + \frac{E}{(\sigma S)}\tilde{\psi} + \frac{E}{(\sigma S)}\hat{\psi} = 0$$ no normal flow

$$w_I + \frac{E}{(\sigma S)^{2/3}}\tilde{w} + \frac{E}{\sigma S \delta_b}\hat{w} = 0$$ no slip on w

Matching (2)

$$\psi_I - \frac{E}{(\sigma S)}\frac{r_o}{2}\tilde{T}_\eta - \frac{E}{(\sigma S)}\frac{r_o}{2}\hat{T}_\xi = 0$$

from

$$\tilde{\psi} = -\frac{r_o}{2}\tilde{T}_\eta \qquad \psi_b = \frac{E}{\sigma S}\hat{\psi}, \qquad \hat{\psi} = -\frac{r_o}{2}\hat{T}_\xi$$

But from thermal wind eqn. and $v_I(r_o, z) + \tilde{v}(0, z) = 0$ for $\sigma S >> E^{2/3}$

Taking z derivative

$$T_{I_r} - \tilde{T}_\eta = 0$$

thus $\hat{T}_\xi = 0$ on $r = r_o$

also $\hat{w} = 0$

Buoyancy layer absent to lowest order
(analogous to slippery Ekman layer)

Matching (3)

thus

$$\psi_I(r_o) - \frac{r_o}{2}\frac{E}{\sigma S}\tilde{T}_\eta = 0,$$

on $r = r_o$

$$\psi_I(r_o) = \frac{E}{\sigma S}\frac{r_o}{2}T_{I_r}$$

Boundary condition expressed now entirely in terms of interior variables.note the streamfnc. is independent of z so T_{ir} must be also independent of z on the rim

using $$r\psi_I(r) = \frac{E^{1/2}}{4}rv_T - \frac{E^{1/2}}{4(1+\lambda)}r\frac{\partial}{\partial r}\int_0^1 \theta dz$$

$$\boxed{\frac{E^{1/2}}{4}r_o v_T - \frac{E^{1/2}}{4}r_o\int_0^1 T_{I_r}dz - \frac{r_o}{2}\frac{E}{\sigma S}\int_0^1 T_{I_r}dz = 0}$$

Side wall boundary condition

With $\dfrac{\partial T_I}{\partial r}(r_o)$ Independent of z we obtain the final boundary condition for the interior thermal equation.

$$T_{I_r}(r_o) = \frac{\lambda}{1+\lambda} v_T(r_o),$$

$$\lambda = \frac{\sigma S}{2E^{1/2}}$$

For the pseudo temperature

$$\boxed{\theta_r(r_0) = \lambda v_T(r_0)}$$

that satisfies

$$\nabla^2\theta = \lambda \frac{1}{r}\frac{\partial}{\partial r}\left(r v_T\right)$$

Example 1:The purely mechanical driven circulation

Suppose (although it is a bit artificial) $T_{I_z} = 0, \quad z = 0,1$

or $\theta_z = 0, \ z = 0,1$

then a solution to $\nabla^2\theta = \lambda \frac{1}{r}\frac{\partial}{\partial r}\left(rv_T\right)$ satisfying $\theta_r(r_o) = \lambda v_T(r_o)$

$$\theta_r(r) = \lambda v_T(r)$$
$$\Rightarrow$$
$$T_{I_r} = \frac{\lambda}{1+\lambda} v_T(r)$$

From the thermal wind equation $v_z = T_r$ $\quad v_I = \frac{\lambda}{1+\lambda} z v_T + V_o(r)$

The mechanically driven velocity

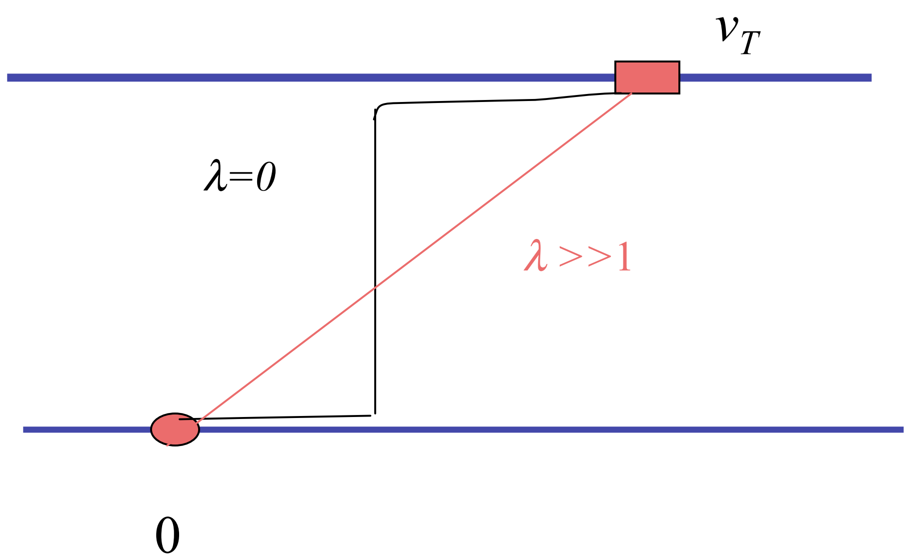
To determine $V_O(r)$ use the boundary condition at $z = 0$

$$w(z=0) = \frac{E^{1/2}}{2}\frac{1}{r}\frac{\partial rv(r,0)}{\partial r} = \frac{E}{2\sigma S}T_{I_{rr}} = \frac{E}{2\sigma S}\frac{\lambda}{1+\lambda}\frac{1}{r}\frac{\partial rv_T}{\partial r} \qquad \lambda = \frac{\sigma S}{2E^{1/2}}$$

$$\Rightarrow V_o = \frac{1}{2(1+\lambda)}v_T$$

$$\boxed{v_I = \frac{\lambda}{1+\lambda}v_T(z-1/2)+v_T/2}$$

v_I as a function of λ



$$w_I = \frac{E^{1/2}}{4(1+\lambda)} \frac{1}{r} \frac{\partial r v_T}{\partial r}$$

$$\lambda = \frac{\sigma S}{2E^{1/2}}$$

For large stratification the interior flow satisfies the no slip boundary conditions on z=0, 1 **expunging the Ekman layers. The Ekman layers are choked off as the stratification increases and the vertical velocity diminishes.** w_I falls from $O(E^{1/2})$ to order $E/\sigma S$ as the stratification increases.

Experiment:
Paul Linden, JFM 1977 **79, 435-448**

Linden used a sugar solution to stratify the fluid. The flow was in an annulus,not a cylinder, but otherwise the theory should apply. The Prandtl number is large σ is about 2.1 10^3

$\sigma S / E^{1/2}$

~5.75 10^4

But σS was large

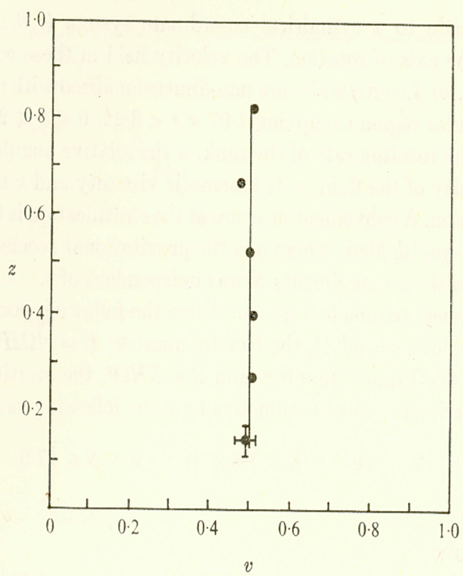


FIGURE 3. The azimuthal velocity v (non-dimensionalized with respect to the velocit lid) at (nominal) mid-radius, plotted against the non-dimensional height, for the unst case $S = 0$. In this case $\epsilon = 0{\cdot}04$ and $E = 5{\cdot}2 \times 10^{-4}$. The solid line represents the velocity as determined from linear theory. The error bars represent the maximum uncertainty in the measurements.

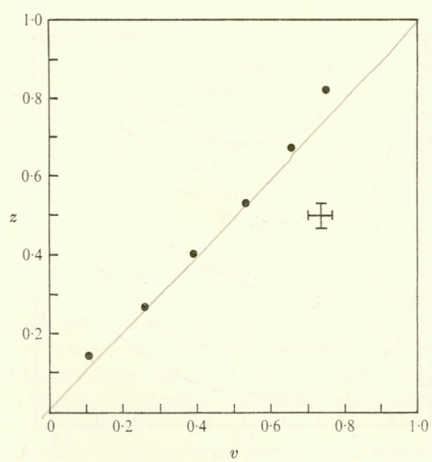


FIGURE 5. As for figure 4 with $S = 0{\cdot}63$, $\epsilon = 1{\cdot}3 \times 10^{-3}$ and $E = 4{\cdot}8 \times 10^{-4}$.

(figure 7) show no evidence of larger shears near the upper and lower boundaries than in the interior. We infer from these measurements that the top and bottom Ekman layers, present in the homogeneous case, are significantly weakened by the stratification, and that for large enough values of S the *interior* zonal velocity satisfies the top and bottom boundary conditions directly. Even for the case

Large Stratification, no Ekman layers

S=0

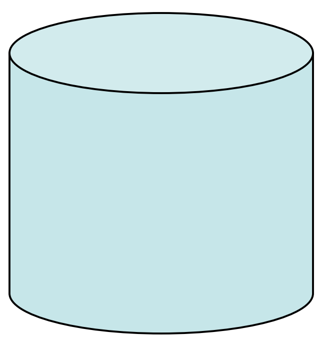
Example 2 The heated cylinder, theory and experiment

Consider the same cylindrical geometry but now the top is *not* moving differentially. Instead the fluid is driven by an applied temperature at the upper boundary.

recall that T refers to the temperature anomaly around the basic stratification.

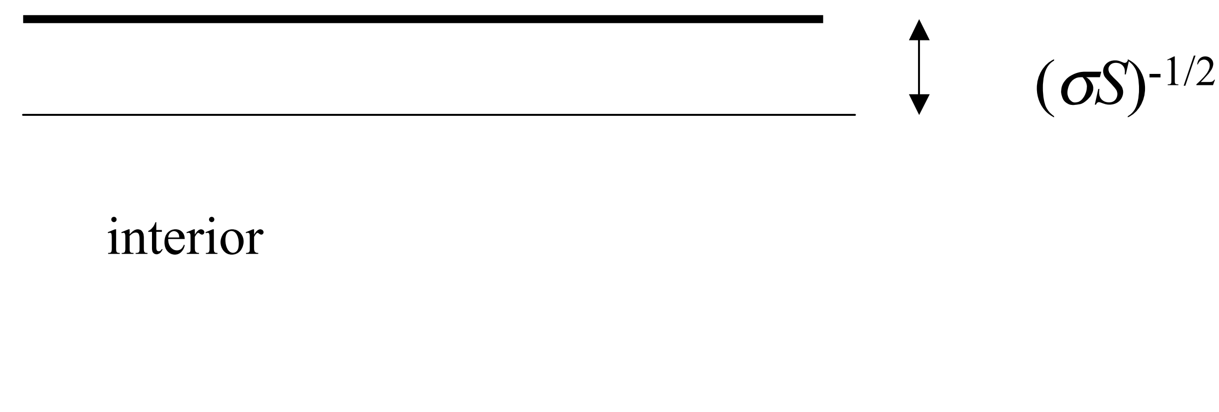
$$T = T_u(r), \qquad z = 1,$$

$$T = 0, \qquad z = 0$$



The $(\sigma S)^{-1/2}$ layer

This system was examined experimentally and spanned a rather wide range of values of S. Indeed, it was rather natural in the laboratory situation to consider values of σS that were large compared to unity as well as small values. When σS is large, the hydrostatic layer has already filled the interior. Indeed, as we shall show below, a metamorphosis of that layer occurs and a boundary layer of scale $(\sigma S)^{-1/2}$ now exists in the vicinity of the upper, heated boundary and it plays a role similar to the $(\sigma S)^{1/2}$ layer on the side wall for small values of S.



Equations of motion

Outside the Ekman layers

$$v = p_r, \qquad u = \frac{E}{2}\left[\nabla^2 v - v/r^2\right],$$

$$T = p_z, \qquad w = \frac{E}{2\sigma S}\left[\nabla^2 T\right]$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$$

$$\nabla^2\left[\sigma S \nabla_h{}^2 + \frac{\partial^2}{\partial z^2}\right] p = 0,$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad \nabla_h{}^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

Balances in the $\sigma S^{-1/2}$ layer

The temperature in this layer is O(1). From the thermal wind equation this means the vertical shear of *v* is order 1

$\longrightarrow$

v is order $\sigma S^{-1/2}$ .This in turn implies that u =O(E(σS)$^{1/2}$) ,w=O(*E*) and the lowest order circulation in the vertical plane is limited to the upper thermal layer for large stratification. The Ekman layers are also, again, absent for large σS.

Below this layer *u* and *w* are so small, in the interior, that the azimuthal equation satisfies,

$$\nabla^2 v_I - v_I / r^2 = 0$$

And so is a Couette flow driven by the condition that the *total v* vanish on both *z* =0 and 1. Details left for the student.

Full system outside boundary layers: $\sigma S >> E^{1/2}$

$$\nabla^2\left[\sigma S \nabla_h{}^2 + \frac{\partial^2}{\partial z^2}\right] p = 0,$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad \nabla_h{}^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

$$\begin{aligned} p_r &= 0, && z = 0,1 \text{ (no Ekman layers)} \\ p_z &= 0, && z = 0, \\ p_z &= T_u(r) && z = 1, \end{aligned}$$

$$p_r = 0, \qquad r = r_o$$

$$p = \sum_{n=1} P_n(z) J_o(k_n r / r_o) \qquad P_n = A_n e^{-q_n(z-1)} + B_n e^{-q_n z} + C_n e^{-q_n d(z-1)} + D_n e^{-q_n dz}$$

$$q_n = k_n / r_o, \quad d = (\sigma S)^{1/2}$$

An idealized problem

suppose $T_u = e^{r/r_o} - 1$

λ= (a) 0.01, (b) 1.0, and (c) 100

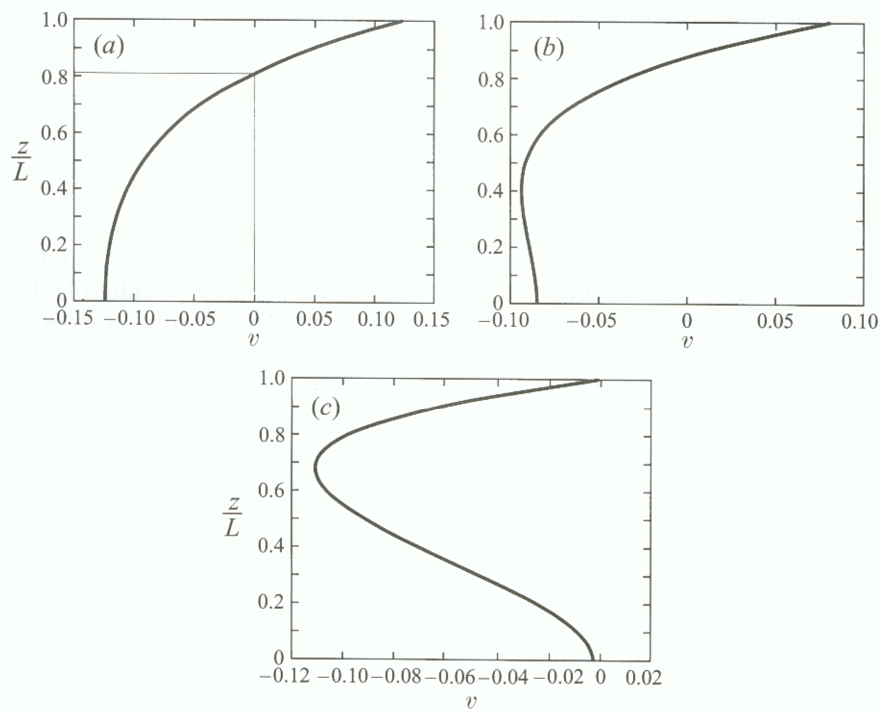


FIGURE 5. The azimuthal velocity profile in the example in which the temperature distribution on the upper plate is $T_u = e^{r/r_o} - 1$. The profile is shown at $r = 0.5r_o$. (a) $\lambda = 0.01$, (b) $\lambda = 1.0$, (c) $\lambda = 100$.

For small and moderate λ the pseudo temperature equation is used.

An experiment

Whitehead and Vetch carried out an experiment in which the upper surface was forced by a temperature anomaly whose *measured* form was

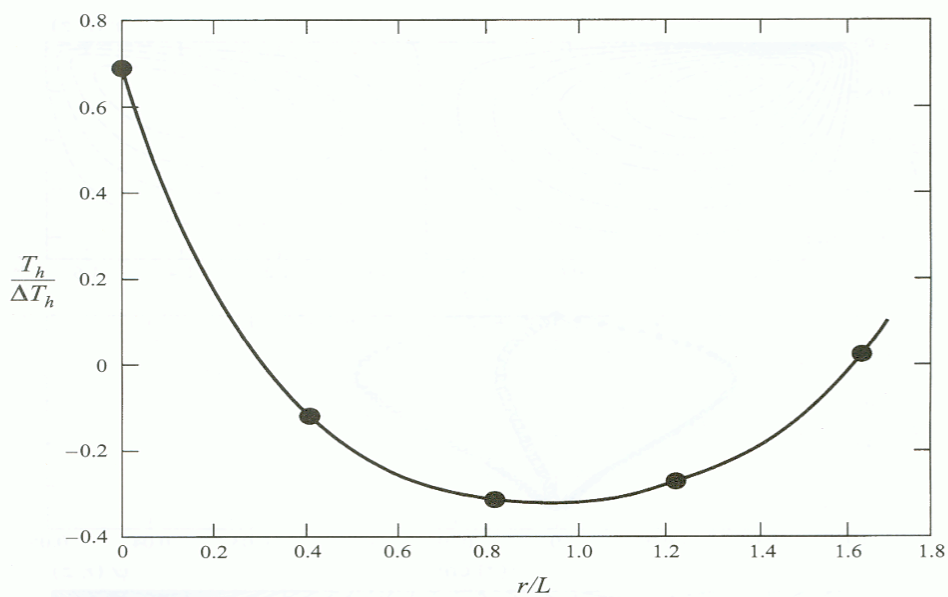


FIGURE 6. The horizontal non-dimensional distribution of temperature at the upper lid of the cylinder for experiment 12/1 of figure 3.

The velocity as seen in dye streaks

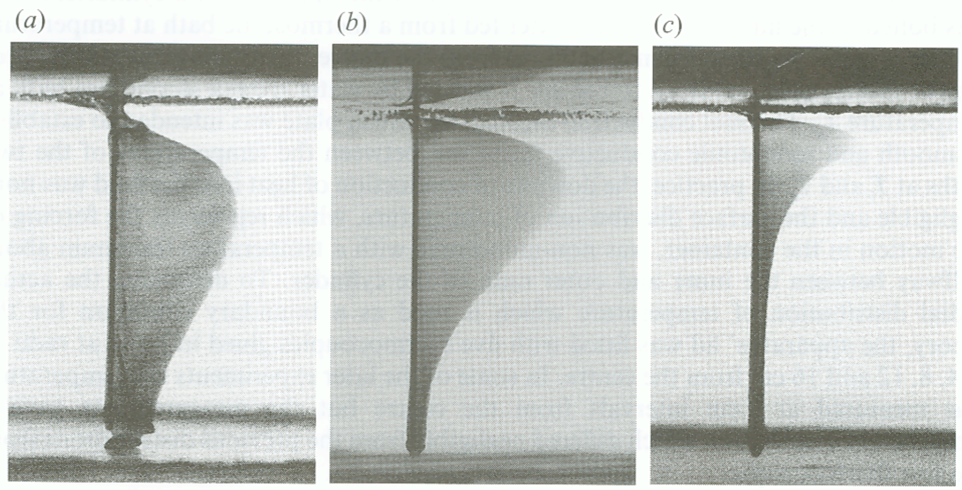


FIGURE 2. The dye streaks demonstrating the profile of azimuthal velocity in the experiment: (a) $S = 0.014$, (b) $S = 0.15$, (c) $S = 29$.

Theoretical predictions

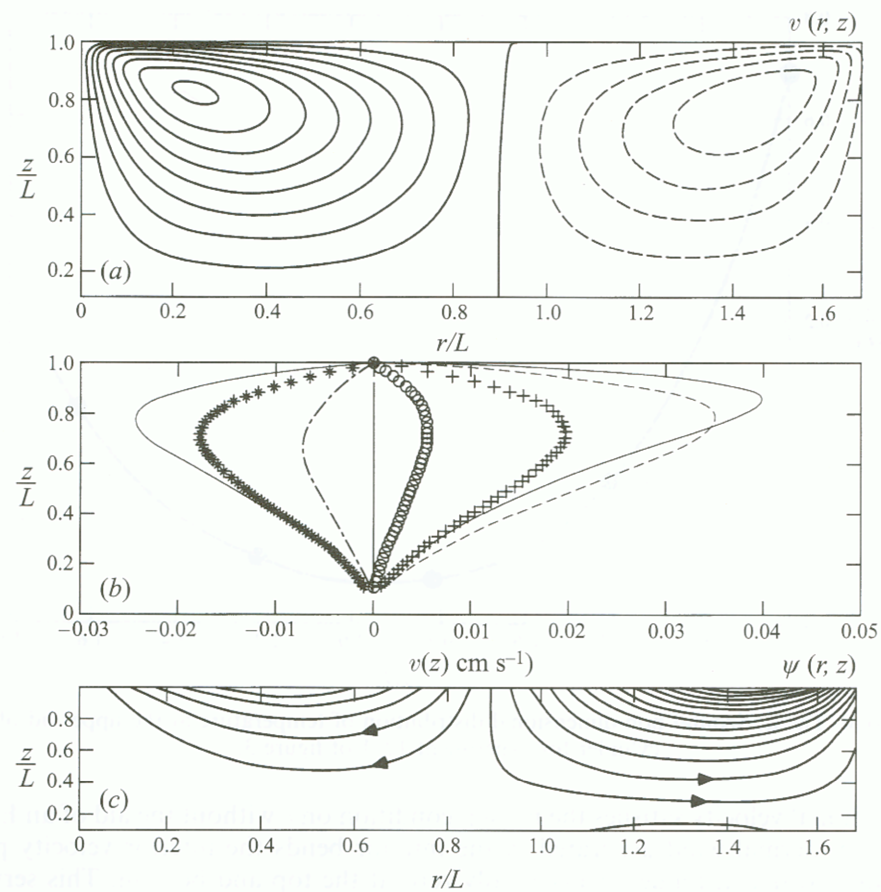


FIGURE 7. The calculated fields of motion for the 12/1 experiment. (*a*) Contours of v in the (r, z)-plane. Each contour interval is $0.005\ \mathrm{cm\ s^{-1}}$. Negative values of v are shown by dashed contours. (*b*) Azimuthal velocity profiles at $r = 2$ (——), $r = 4$ (– –), $r = 6$ (+), $r = 8$ (○), $r = 10$ (–·–), $r = 12$ (∗), $r = 14$ (solid). (*c*) Contours of ψ.

Comparison with experiment

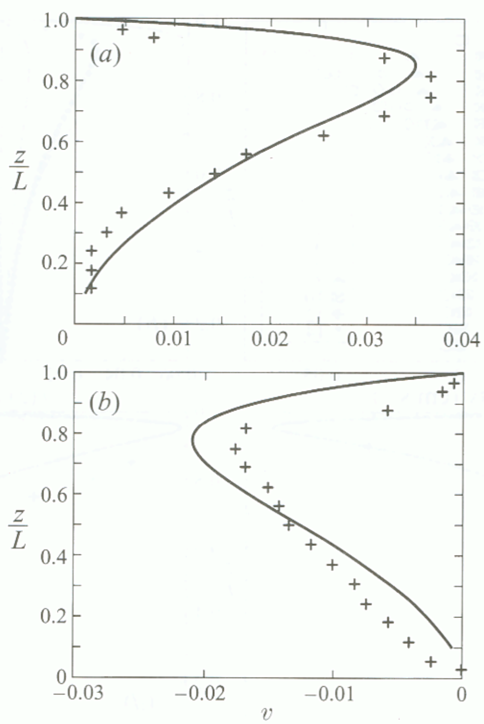


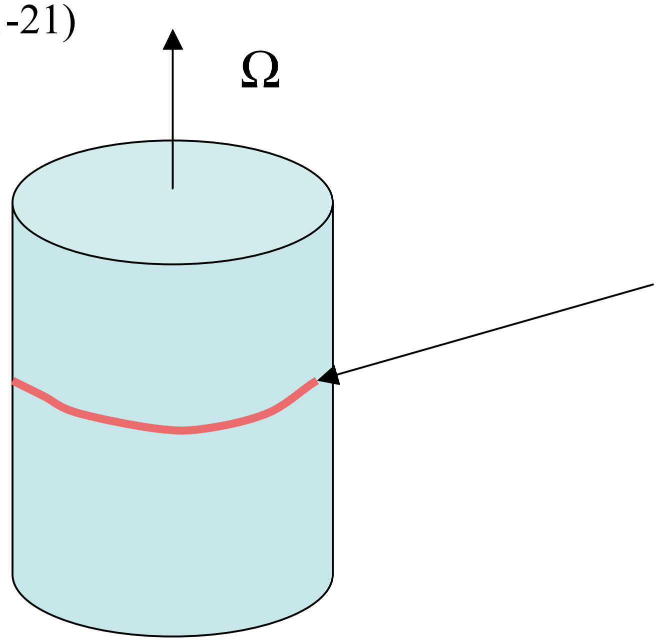
FIGURE 8. A direct comparison of theory and experiment for two profiles for experiment 12/1 for which the stratification is moderate. The experimental data are indicated by + symbols. (*a*) $r = 2$ cm, (*b*) $r = 14$ cm.

$\lambda = 42$

Good agreement

Example 3: Side wall heating:driven by buoyancy layer

We consider now a case in which the flow is driven by the side wall buoyancy layer. (Whitehead, J.A. and J. Pedlosky 2000, Circulation and boundary layers in differentially heated rotating stratified fluid. *Dyn. Atmospheres and Oceans*, **31**, 1-21)



heating coil

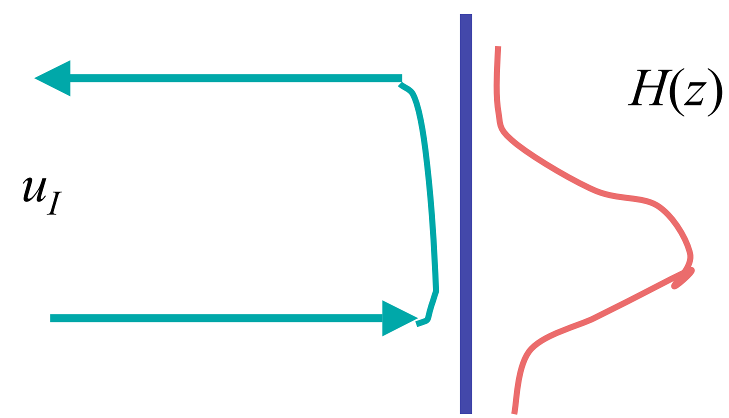
$H(z)$ is shape of heating function on side wall

The buoyancy layer suction

$$\hat{\psi} = \frac{E}{\sigma S}\frac{r_o}{2}H(z)e^{-\xi}\left[\sin\xi + \cos\xi\right]$$

$$\longrightarrow \quad \psi_I = -\frac{E}{\sigma S}\frac{r_o}{2}H(z)$$

$$\Rightarrow u_I(r_o, z) = -\frac{1}{r_o}\frac{\partial \psi_I}{\partial z} = \frac{E}{2\sigma S}\frac{dH}{dz}$$



this circulation will drive, via the Coriolis
torque a strong azimuthal velocity

Equations and boundary conditions

$$\nabla^2\left[\sigma S \nabla_h{}^2 + \frac{\partial^2}{\partial z^2}\right] p = 0,$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad \nabla_h{}^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

from

$$v = p_r, \qquad u = \frac{E}{2}\left[\nabla^2 v - \frac{v}{r^2}\right],$$ and $v = p_r = 0$ on r_o

$$\frac{\partial}{\partial r}\nabla_h{}^2 p = \frac{1}{\sigma S}\frac{\partial H}{\partial z}, \qquad r = r_o$$

on z =0,1

$$\nabla^2 p_z = \mp \frac{\sigma S}{E^{1/2}} \nabla_h{}^2 p, \qquad \nabla_h{}^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

Again, solution obtained by a Bessel expansion in *r*.

Results of side heating

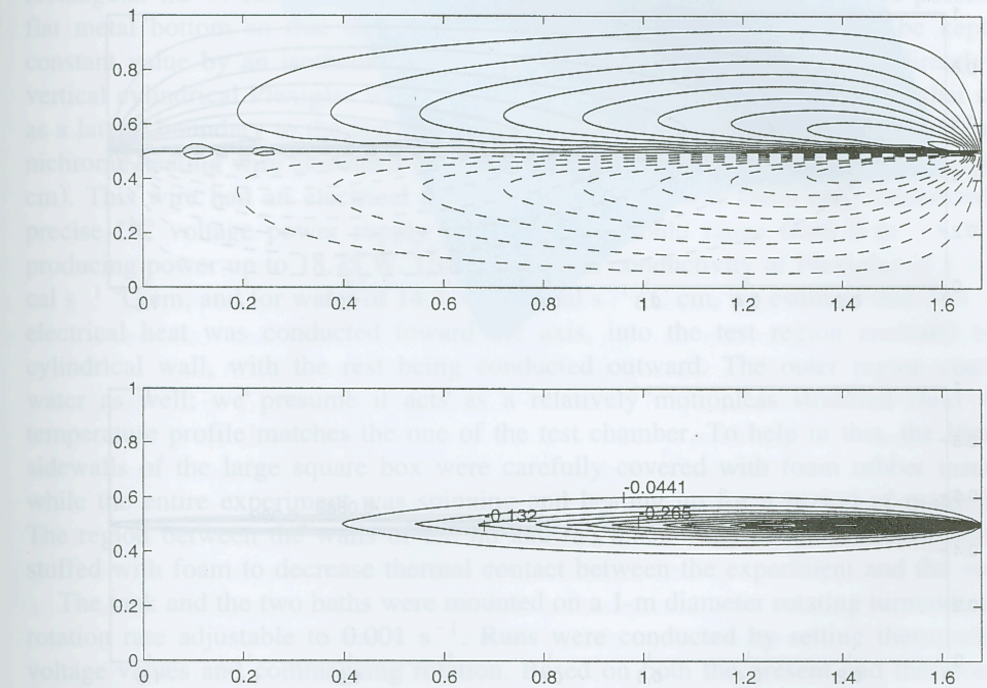


Fig. 4. Top: Azimuthal velocity produced by a point heater located midway along the side boundary. Bottom: Meridional stream function from a point heater. $S = 13.44$; $E = 7.5 \times 10^{-4}$.

Contours of v

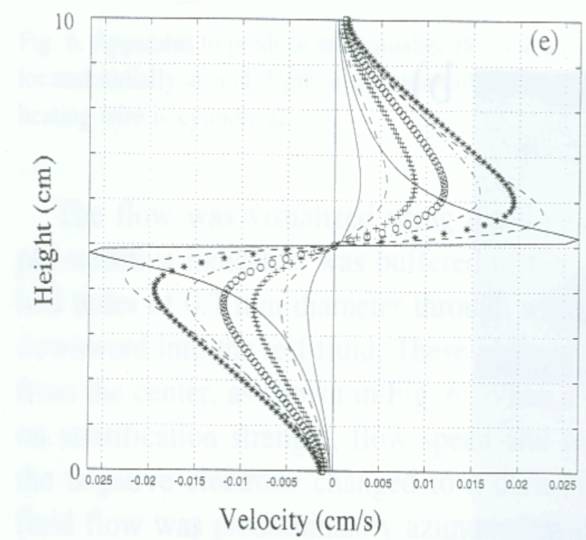
Contours of ψ

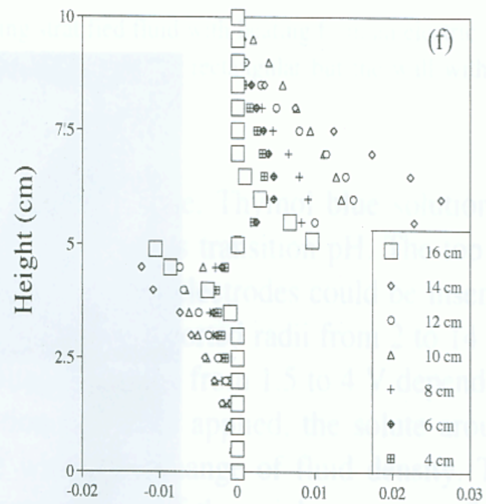
S= 13.44, E=7.5 10^{-4}

Velocity profiles v(z) at various radii

Theory

Experiment





S= 13.44, E=7.5 10^{-4}

Velocity profiles, larger stratification

Theory / Experiment

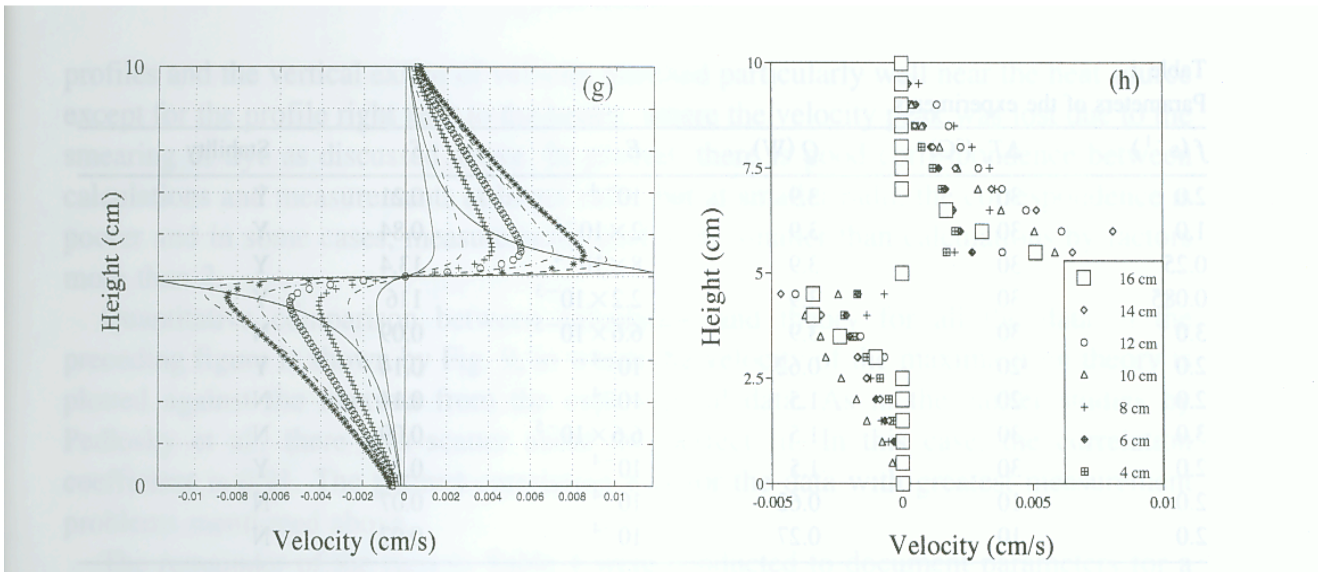


Fig. 8 (continued).

S = 116.2 and $E = 2.2\ 10^{-3}$

Further comparison

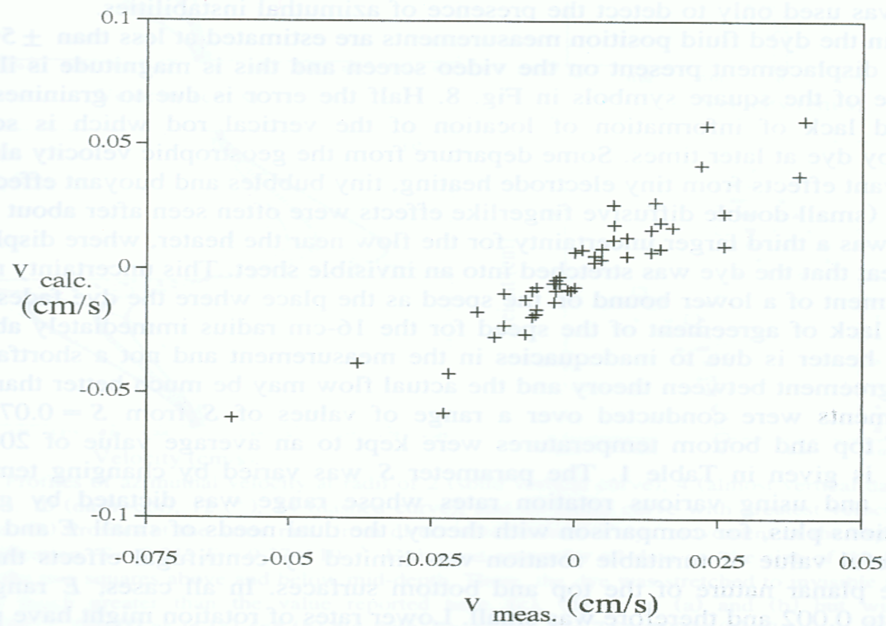


Fig. 9. Calculated and measured velocity maxima from the profiles shown in Fig. 8.

A comparison of the maximum azimuthal velocity from 4 parameter setting between theory and experiment. See Whitehead &Pedlosky (2000) for details

Further thoughts on the role of stratification.

The Ekman layers and the side wall boundary layers combine to set up a vertical circulation in the interior that is closed in the side wall layers for small *S*.

In our cylinder example it is a two stage process. First the flux in the Ekman layer forces vertical motion in the interior and also feeds the side wall layers.

Then the side wall layers flux fluid vertically and establish a heating and cooling of the interior altering the strong, O(1) circulation through the thermal wind balance. For large *S* this can choke off the bottom Ekman layers and the frictional dissipation there.

If the side walls and the bottom should combine, i.e. should the bottom be sloping with respect to the stratification, this 2 step program could coalesce.