

GFD 2007

Boundary Layers: Homogeneous Ocean Circulation

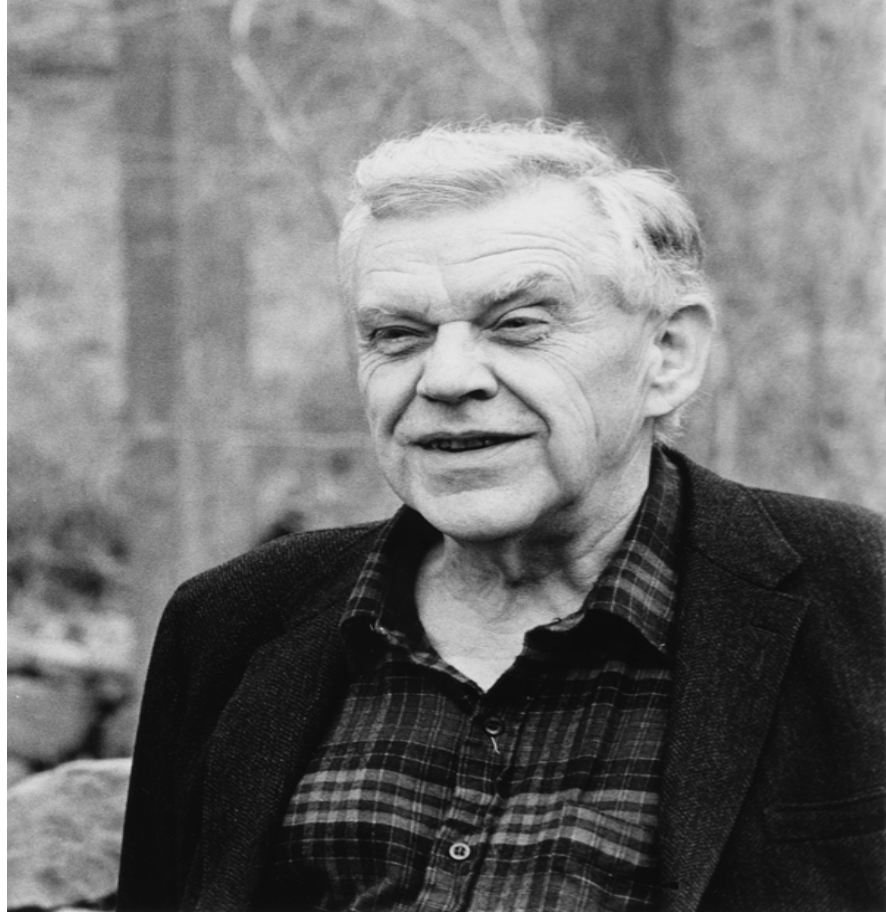
One of the most significant applications of boundary layer theory occurs in the treatment of the oceanic general circulation

Stommel, H. 1948 The westward intensification of oceanic currents. *Trans. Amer. Geophys. Union*, 29, 202-206

¹Munk, W. 1950 The wind-driven ocean circulation. *Met.* 7, 73

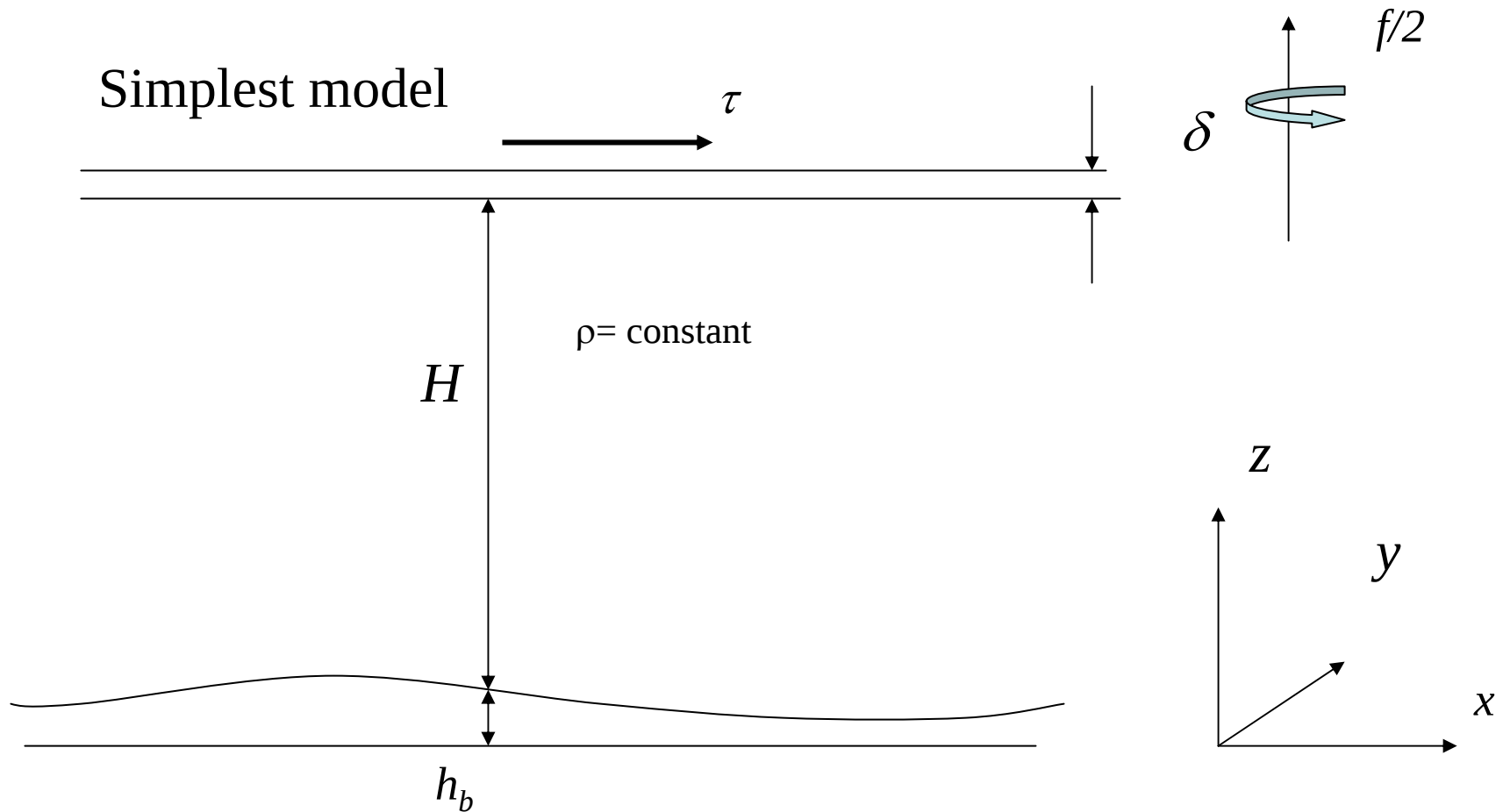
Munk W.H. and G.F. Carrier, 1950 The wind-driven circulation in basins of various shapes. *Tellus*, 2, 158-167.

Henry Stommel



Holly Pedlosky

The homogeneous model



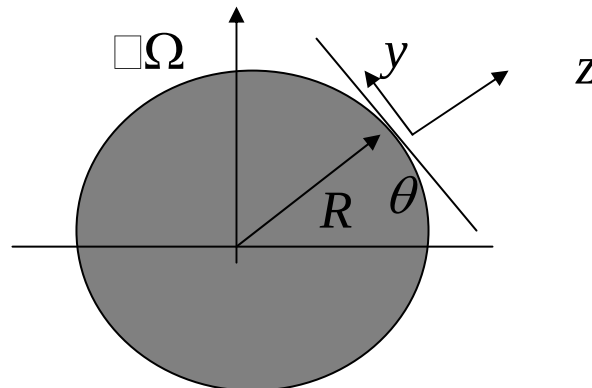
The model

Upper Ekman layer provides a pumping

$$w = w_e = \hat{k} \mathbf{g} \nabla \times (\boldsymbol{\tau} / \rho f) \quad z = H + h_b$$

$$w = \frac{\delta}{2} \zeta + \mathbf{u} \mathbf{g} \nabla h_b = \frac{\delta}{2} [v_x - u_y] + u h_{bx} + v h_{by} \quad z = h_b$$

β plane



$$f_o = 2\Omega \sin \theta$$

$$f = f_o + \beta y, \quad \beta = \frac{\partial f}{\partial y} = \frac{2\Omega \cos \theta}{R}$$

Model equations of motion (1)

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v = f\frac{\partial w}{\partial z} + A\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta$$

Vorticity equation

Integrating vertically,

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v + \frac{fu}{H}\frac{r}{g}h_b = \frac{fw_e}{H} - \frac{f\delta}{2H}\zeta + A\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta$$

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}, \quad \psi = \frac{p}{\rho f_0}$$

Geostrophic stream function

Equations of motion (2) and scaling

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) + \beta \psi_x + \frac{f}{H} J(\psi, h_b) = \frac{f w_e}{H} - \frac{f \delta}{2H} \nabla^2 \psi + A \nabla^4 \psi$$

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}, \quad \psi = p / \rho f_o$$

U , L for velocity and length, UL for ψ

$(\beta L)^{-1}$ or time. Choose U as

$$U = \frac{\tau_o}{\rho H_o L \beta}$$

Ekman pumping scales with

$$W_e = \tau_o / \rho f L$$

$$\frac{\partial}{\partial t} \nabla^2 \psi + \delta_l^2 J(\psi, \nabla^2 \psi) + \psi_x + \eta J(\psi, h_b) = w_e - \delta_s \nabla^2 \psi + \delta_m^3 \nabla^4 \psi$$

Governing equation and boundary layer scales

$$\frac{\partial}{\partial t} \nabla^2 \psi + \delta_I^2 J(\psi, \nabla^2 \psi) + \psi_x + \eta J(\psi, h_b) = w_e - \delta_s \nabla^2 \psi + \delta_m^3 \nabla^4 \psi$$

$$\delta_I = \frac{(U/\beta)^{1/2}}{L}, \quad \eta = \frac{f \Delta h_b}{H_o \beta L}, \quad \delta_s = \frac{f \delta}{2 H_o \beta L}, \quad \delta_m = \frac{(A/\beta)^{1/3}}{L}$$

Inertial
scale

Stommel

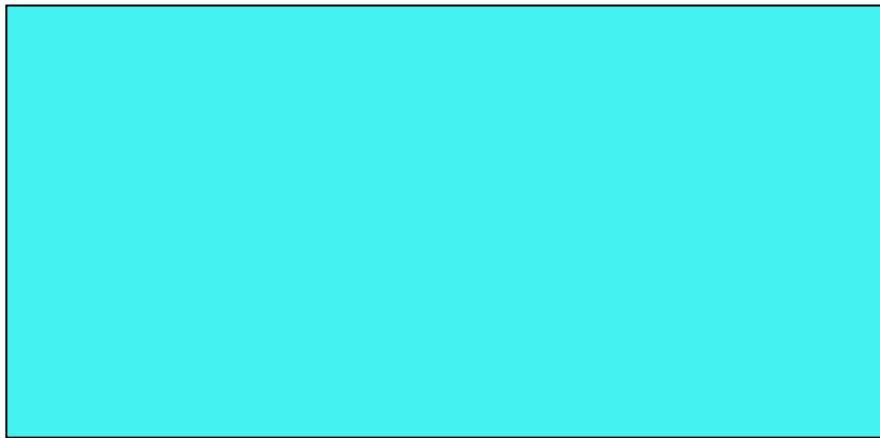
Munk

Relative strength of bottom
topography to β effect

The singular perturbation problem

δ_I , δ_M , and δ_S are all small. Boundary layer scales are much less than the full basin width. They multiply the higher order derivatives.

$y = 1$



$x = 0, y = 0$

$x = x_e$

L is the north-south extent of the basin.

The interior problem

For a flat bottom *interior*, when all the boundary layer scales are small, the governing equation is :

$$\psi_x = w_e(x, y)$$

Sverdrup relation. 1st order ode in x alone. Only determines interior meridional velocity.

Can't satisfy no slip and can satisfy no normal flow, or $\psi = 0$, only on one boundary, east or west but not both.

Two possible (at least) interior solutions

1) Satisfy $\psi = 0$ on western boundary

$$\psi = \int_0^x w_e(x', y) dx'$$

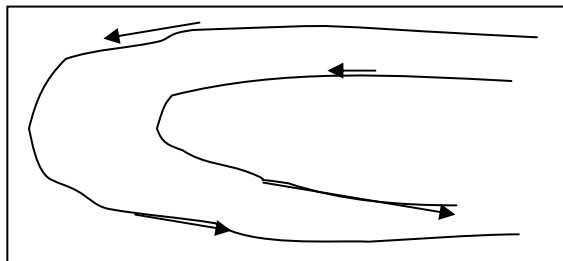
or 2) on eastern boundary

$$\psi = - \int_x^{x_e} w_e(x', y) dx'$$

example

$$\psi_1 = (x_w - x) \sin \pi y,$$

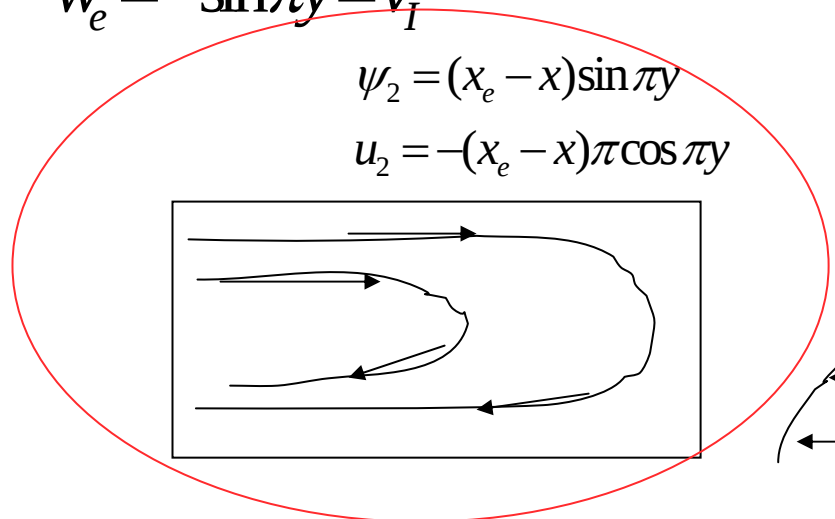
$$u_1 = -(x_w - x) \pi \cos \pi y$$



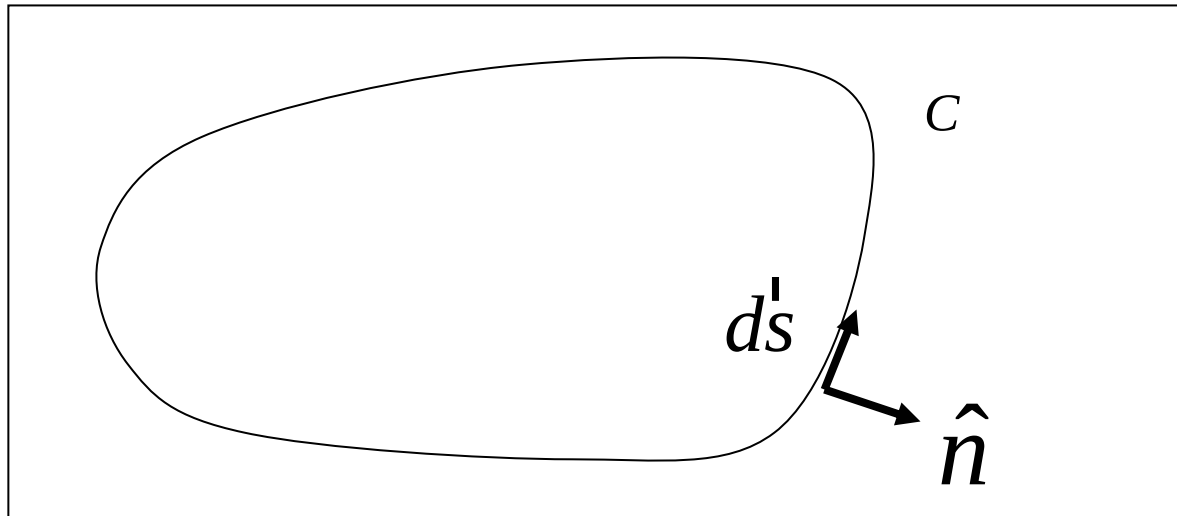
$$w_e = -\sin \pi y = v_I$$

$$\psi_2 = (x_e - x) \sin \pi y$$

$$u_2 = -(x_e - x) \pi \cos \pi y$$



An Integral constraint (1)



C a steady (closed) streamline.

$$\oint_C \frac{\partial u}{\partial t} \hat{g} ds + \oint_C \left[\frac{1}{2} \delta^2 \zeta + y + \eta_b \right] \hat{g} ds = \oint_C \frac{1}{2} \omega \hat{g} ds - \delta_s \oint_C \frac{1}{2} \omega \hat{g} ds + \delta_m^3 \oint_C \nabla \cdot \hat{g} ds$$

= 0

An Integral Constraint (2)

$$0 = \int_C \tau \hat{g} ds - \delta_s \int_C u \hat{g} ds + \delta_m^3 \int_C \nabla \zeta \hat{g} ds$$

The net input of vorticity on *each streamline* must be balanced by bottom friction and lateral friction (for steady flow)

If there are eddies that flux vorticity integral must include that effect.

$$0 = \int_C \tau \hat{g} ds - \delta_s \int_C u \hat{g} ds + \delta_m^3 \int_C \nabla \zeta \hat{g} ds - \delta_I^2 \int_C \overline{u' \zeta'} \hat{g} ds$$

but this last term must be zero for the streamline coincident with boundary.

The Energy constraint

Multiplying by ψ and integrating over the closed basin for steady flow:

$$\langle w_e \psi \rangle = -\delta_s \langle |\nabla \psi|^2 \rangle - \delta_m^3 \langle |\nabla^2 \psi|^2 \rangle$$

So that ψ and w_e must be negatively correlated. On the whole this implies a circulation in the direction of the wind stress.

The linear boundary layer problem

$$\delta_I \ll \delta_s, \delta_m$$

For a flat bottom, linear eqn. becomes .

$$\psi_x = w_e - \delta_s \nabla^2 \psi + \delta_m^3 \nabla^4 \psi$$

and interior solution is

$$\psi_I(x,y) = - \int_x^{x_e} w_e(x',y) dx' + \Psi(y)$$

The Stommel model

In the boundary layer, keeping only x derivatives and letting $x = \delta \xi$

After a single integration in x

$$\phi = -(\delta_s / \delta) \frac{\partial \phi}{\partial \xi} + \left(\frac{\delta_m^3}{\delta^3} \right) \frac{\partial^3 \phi}{\partial \xi^3}$$

Consider case

$$\delta_m \ll \delta_s$$

Ignore term c (this is a singular perturbation) to obtain Stommel's model of the boundary layer.

$$\frac{\partial \phi}{\partial \xi} + \phi = 0$$

The Stommel solution

$$\phi = A(y)e^{-\xi}$$

To satisfy no normal flow
condition on $x=0$

$$A = -\psi_I(0, y)$$

If we try to do the same on the *eastern* boundary $\xi' = (x_e - x) / \delta_s$

$$\phi = \frac{\partial \phi}{\partial \xi'}$$

bl correction function grows exponentially. **No
boundary layer possible on eastern boundary.**

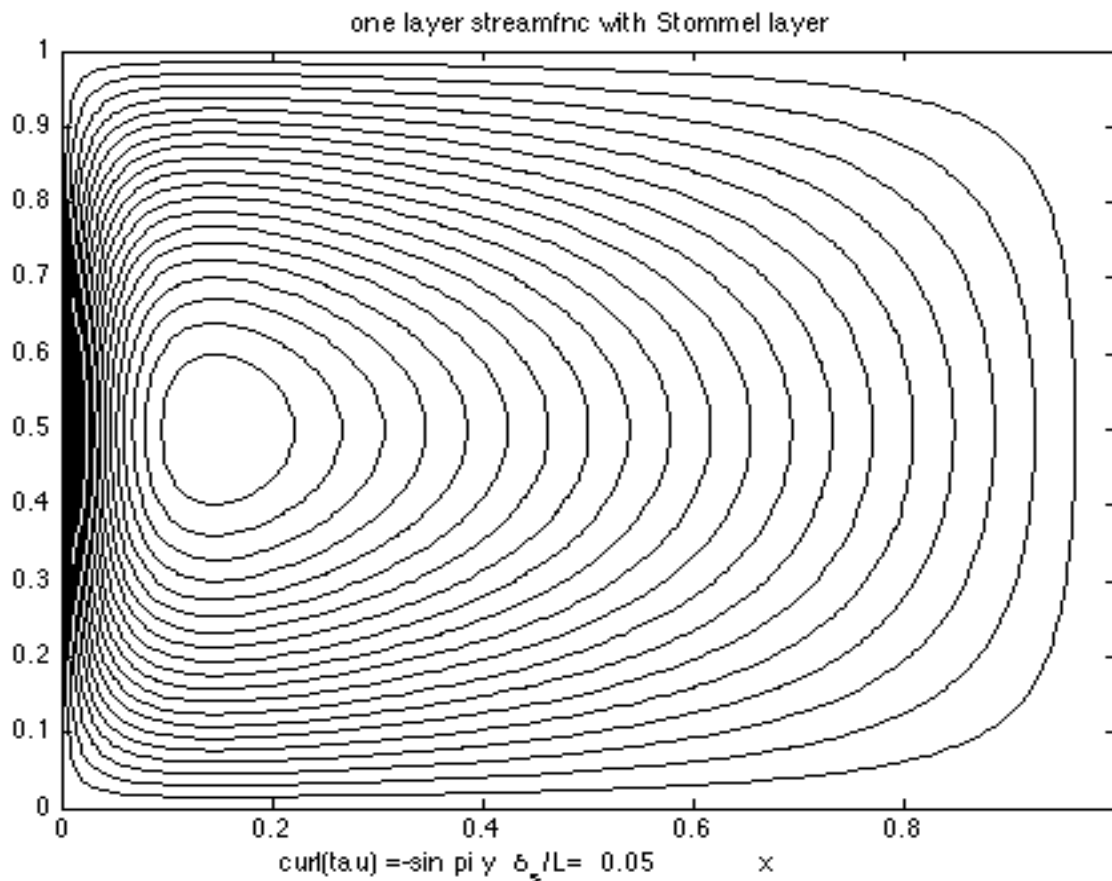
Hence, $\psi(y) = 0$ in interior solution

$$\psi(x, y) = \psi_I(x, y) + \psi_I(0, y)e^{-x/\delta_s},$$

$$\delta_s = f\delta / 2H\beta L$$

$$\psi_I(x, y) = - \int_x^{x_e} w_e(x', y) dx'$$

The western intensification



Stommel's original explanation of western intensification and the existence of the Gulf Stream due to β effect.

Controlled by boundary layer

The no slip condition and the sublayer

- Need to satisfy no slip condition .So far ignored.
- The vorticity balance of the whole basin depends on the lateral diffusion term if no slip condition applies. So far ignored.

To preserve the total order of the system and to satisfy the no slip condition we need to include terms b and c in boundary layer equation. Now, $x = \delta_s \xi$

$$\frac{\partial \phi}{\partial \xi} = -\frac{\partial \phi}{\partial \xi} + \left(\frac{\delta_m^3}{\delta_s^3} \right) \frac{\partial^3 \phi}{\partial \xi^3} \quad \text{For sub layer define } \xi = \eta$$

$$\delta_{sub} = \delta_s \ell$$

$$\ell = \left(\frac{\delta_m}{\delta_s} \right)^{3/2} = \frac{\delta_m^{3/2}}{\delta_s^{1/2}}$$

Correction function in sublayer

$$\chi(\eta)$$

$$\chi_{\eta\eta} - \chi = 0$$

Essentially, the
Stewartson $E^{1/4}$ layer.
Independent of β ,
symmetric east west.

$$\chi = C e^{-\eta}$$

$$\delta_{sub} = \left[\frac{A}{L^2} \frac{H_o}{\sqrt{2\nu f}} \right]^{1/2}$$

$$\psi = \psi_I(x, y) + A(y)e^{-x/\delta_s} + C(y)e^{-x/\delta_{sub}}$$

matching

$$\psi(0, y) = 0 = \psi_I(0, y) + A(y) + C(y)$$

$$\psi_x(0, y) = 0 = \psi_{I_x}(0, y) - \frac{A}{\delta_s} - \frac{C}{\delta_{sub}}$$

Total solution (linear)

$$\delta_s \ll 1, \quad \delta_{sb} \ll \delta_s$$

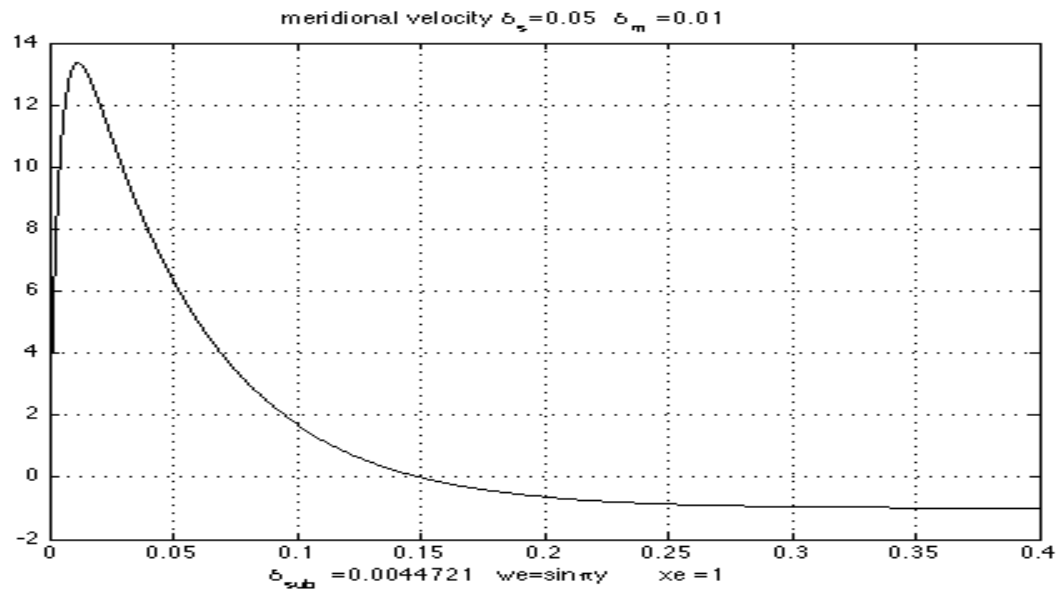
$$C = -\frac{\delta_{sub}}{\delta_s} A,$$

$$A = -\psi_I(0, y)$$

$$\Rightarrow \psi = \psi_I(x, y) - \psi_I(0, y) \left[e^{-x/\delta_s} - \left(\frac{\delta_m}{\delta_s} \right)^{3/2} e^{-x/\delta_{sub}} \right],$$

$$v = \frac{\partial \psi_I(x, y)}{\partial x} + \frac{\psi_I(0, y)}{\delta_s} \left[e^{-x/\delta_s} - e^{-x/\delta_{sub}} \right]$$

Velocity profile in boundary layer near western boundary



The dissipation balance (1)

Integrate across basin. Ignore y derivatives in dissipation terms. For Ekman pumping independent of x ,

$$0 = \cancel{\chi_e w_e} + \cancel{\delta_s \psi_x(0)} - \delta_m^3 \psi_{xxx}(0) \quad \text{Boundary current has no net vorticity}$$

=0 for no slip

The contribution of the sublayer to the final term is:

$$-\chi_e w_e \delta_m^3 \left(\frac{\delta_m}{\delta_s} \right)^{3/2} \frac{1}{\delta_s^3 \left(\frac{\delta_m}{\delta_s} \right)^{9/2}} = -\chi_e w_e$$

Balances input of vorticity

The dissipation balance (2)

Most of the fluid flowing south in the interior returns in the Stommel layer and *not* the sublayer. On those streamlines *always outside the sublayer* the dissipation balance only involves bottom friction,

Integrating across the basin from just outside the sublayer to the eastern boundary, the total mass flux balances and:

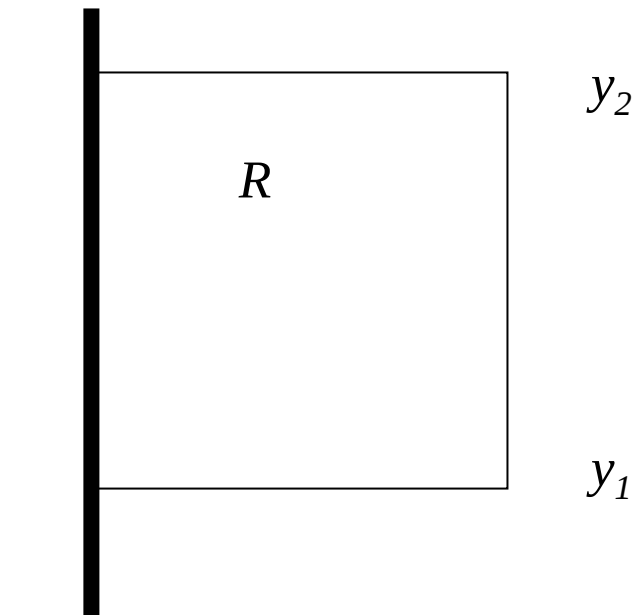
$$0 = x_e w_e(0_+, y) + \delta_s \phi_x(0_+, y),$$

$\Rightarrow$

$$0 = x_e w_e(0_+, y) - \delta_s x_e w_e(0, y) / \delta_s$$

Vorticity balance on streamlines
through Stommel layer

An integral balance for the boundary layer



Ignoring vorticity input by
wind *in the boundary layer*
region

$$x = 0 \quad \int_R \delta_I^2 \nabla \mathbf{g} \cdot \zeta dA + \int_R v dA + \eta \int_R \nabla \mathbf{g} \cdot \mathbf{h}_b dA = -\delta_s \int_R \zeta dA + \int_R \delta_m^3 \nabla^2 \zeta dA$$

The integral balance with bl approximations

$$\delta_I^2 \frac{1}{2} [v^2(0, y_1) - v^2(0, y_2)] + \int_{y_1}^{y_2} \psi_I(0, y) dy - \eta \left[\overline{\psi \frac{\partial b}{\partial x}}_{y=y_2} - \overline{\psi \frac{\partial b}{\partial x}}_{y=y_1} \right]$$

used

$$\zeta \approx v_x$$

$$= \delta_s \int_{y_1}^{y_2} v(0, y) dy - \delta_m^3 \int_{y_1}^{y_2} \zeta_x(0, y) dy$$

If the bottom is flat and the no slip condition applies

$$\int_{y_1}^{y_2} \psi_I(0, y) dy = -\delta_m^3 \int_{y_1}^{y_2} \zeta_x(0, y) dy$$

the vorticity put into the fluid along latitude y must be dissipated in the boundary layer at that latitude to obtain a steady state balance. In the presence of an uneven bottom the pressure drag can locally enter the balance but when integrated along a closed streamline the topographic term can give no net contribution (just as the planetary or relative vorticity advection)

See Hughes C. W. and B. de Cuevos. 2001 Why western boundary currents in realistic oceans are inviscid: A link between form stress and bottom pressure torques. *J.Phys. Ocean.* **31**, 2871-2885.

Inertial boundary layers

$$\delta_I \gg \delta_m \gg \delta_s$$

Most fluid will go through inertial layer but there is not enough dissipation in the layer to satisfy the vorticity balance on those streamlines

$$\xi = x / \delta_I$$

$$\psi_{\xi} \psi_{\xi\xi} - \psi_y \psi_{\xi\xi\xi} + \psi_{\xi} = 0$$



$$\psi_{\xi\xi} + y = Q(\psi)$$

Total vorticity conserved on streamline

Inertial boundary layer: Example



$U = \text{constant}$

$$\psi_{\xi\xi} + y = Q(\psi)$$

Far from the boundary the relative vorticity is negligible so

$$\xi \longrightarrow \infty$$

$$Q(\psi) \longrightarrow y$$

$$\psi \longrightarrow y$$



$$Q(\psi) = \psi$$

*On all streamlines
connected to far field*

Inertial layer (2)

$$\psi_{\xi\xi} + y = \psi$$

$$\psi = y[1 - e^{-\xi}]$$

In non-dimensionless units

$$\psi_* = Uy \left[1 - \exp \left\{ -x \left(\frac{\beta}{U} \right)^{1/2} \right\} \right]$$

Interior flow needs to be westward.

Can't close circulation
or satisfy no slip.

Greenspan H.P. 1962 A criterion for the existence of inertial boundary layers in the oceanic circulation. *Proc. Nat. Acad.Sci.*, **48**, 2034-2039.

Pedlosky, J. 1965 A note on the western intensification of the oceanic circulation. *J. Marine Res.* , **23**, 207-209.

Inertial sublayer

$$\delta_d = \left(\frac{A\delta_{*I}}{U} \right)^{1/2} = \delta_{*I} \left(\frac{\delta_m}{\delta_I} \right)^{3/2}$$

vorticity flux through sublayer
could balance vorticity input by
the wind.

Most streamlines don't go
through sublayer.

In Stommel model the streamlines that did not go through the sublayer still had a proper vorticity balance. This is no longer true.

$$R_e = \left(\frac{\delta_I}{\delta_m} \right)^3 \gg 1$$

Boundary layer Reynolds number

Inertial/viscous in inertial layer

What happens?

Inertial Runaway

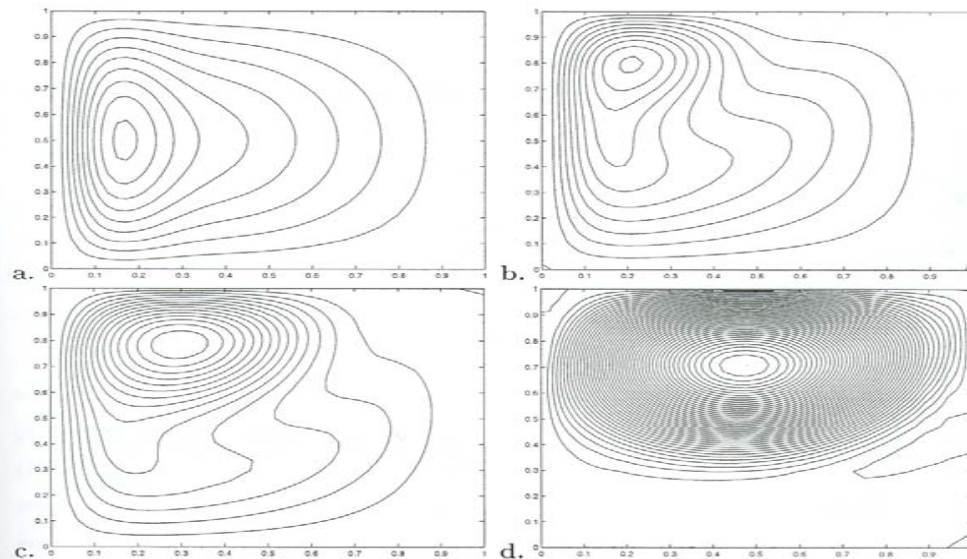


Figure 1-2: (a-d) contour the steady-state streamfunction for the homogeneous model with lateral friction only. The frictional boundary conditions are no-slip on the eastern and western boundaries and slip on the others. (a) shows the $\delta_I = 0$, $\delta_M = 0.05$ calculation and has $\max(\psi) = 0.93$. (b) shows the $\delta_I = 0.05$, $\delta_M = 0.05$ calculation and has $\max(\psi) = 1.02$. (c) shows the $\delta_I = 0.0625$, $\delta_M = 0.05$ calculation and has $\max(\psi) = 1.5$. (d) shows the $\delta_I = 0.08125$, $\delta_M = 0.05$ calculation and has $\max(\psi) = 4.5$. The contour interval is 0.1 in units where 1 is the maximum of the Sverdrup solution.

Panel a shows the linear solution when δ_I is zero, panel b shows the case for $\text{Re}=1$, panel c shows the flow for $\text{Re} = 1.95$

$$\delta_I = .0625, \delta_m = .05$$

, while for panel d, $\text{Re}=4.29$,

$$\delta_I = .08125, \delta_m = .05$$

Circulation intensifies until vorticity is dissipated on each streamline.

References

Veronis, G. 1966 Wind-driven ocean circulation-part I. Linear theory and perturbation analysis. *Deep-Sea Res.* **13**, 17-29

Ierley, G.R. and V.A. Sheremet. 1995 Multiple solutions and advection-dominated flows in the wind-driven circulation. Part I: Slip. *J. Marine Res.* **53**, 703-737,

Fox-Kemper, B. 2003. Eddies and Friction: Removal of vorticity from the wind-driven gyre. *MIT/WHOI Joint Program Ph.D. thesis*

Fox-Kemper, B. and J. Pedlosky, 2004. Wind-driven barotropic gyre I: Circulation control by eddy vorticity fluxes to an enhanced removal region. *J. Marine Res.*, **62**, 169-193.

The enhanced sublayer

When we set a value of A , the momentum mixing coefficient, we are conflating two somewhat independent physical processes.

The first is a measure of the unresolved eddy scales and their effect on the large scale flow in the interior and the boundary layers.

The second is a measure of the strength of the interaction of the fluid with the boundary.

Decreasing the single parameter, A , then reduces both processes. If the interaction with the boundary is related to a different physical process than the dissipation of vorticity away from the boundary, it seems overly constraining to represent both with a single parameterization.

Fox-Kemper allowed the dissipation to locally increase in a sublayer near the western boundary.

$$\delta_m^3 = \frac{\delta_I^3}{\text{Re}_i} + \left(\frac{\delta_I^3}{\text{Re}_b} - \frac{\delta_I^3}{\text{Re}_i} \right) \left(e^{-x/\delta_d} + e^{-(1-x)/\delta_d} \right) \quad \text{Re}_i = \left(\frac{\delta_I}{\delta_m} \right)^3_{\text{interior}} \quad \delta_d = \frac{\delta_I}{\sqrt{\text{Re}_i}}$$

The enhanced sublayer (2)

$$\delta_m^3 = \frac{\delta_I^3}{\text{Re}_i} + \left(\frac{\delta_I^3}{\text{Re}_b} - \frac{\delta_I^3}{\text{Re}_i} \right) \left(e^{-x/\delta_d} + e^{-(1-x)/\delta_d} \right)$$

$$\delta_m^3 = \frac{\delta_I^3}{\text{Re}_b} \quad \text{Near the boundary}$$

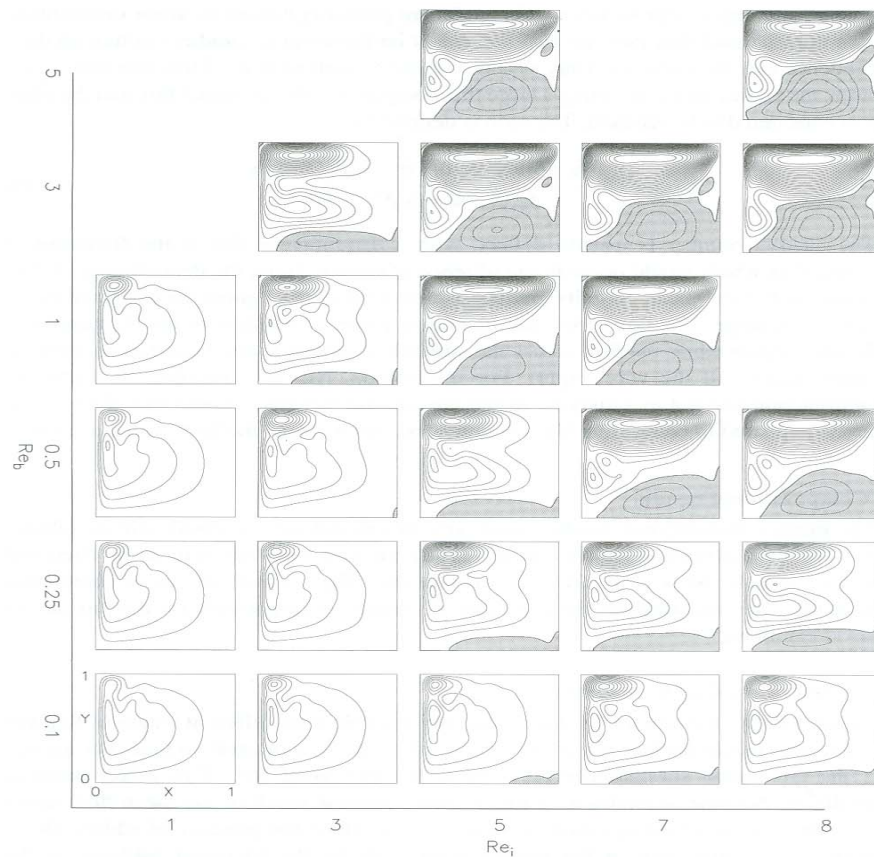
$$\frac{\partial}{\partial t} \nabla^2 \psi + \delta_I^2 J(\psi, \nabla^2 \psi) + \psi_x + \eta I(\psi, h_b) = w_e - \delta_s \nabla^2 \psi + \nabla g \{ \delta_m^3 \nabla(\nabla^2 \psi) \}$$

↑
variable

The turbulent boundary layer and the role of eddies.

For substantial values of the interior Reynolds numbers the western boundary layer becomes unstable to shear instability. The eddies in the inertial portion of the boundary layer, through which most of the mean streamlines pass, will flux vorticity to the sublayer where it is dissipated by the locally enhanced friction. The process is a three step one, instability, flux and dissipation and the student is referred to Fox-Kemper and Pedlosky for the details of the analysis. The result though is striking and shown in the following figure.

The arrested runaway



compare

Control of large
scale circulation by
details of dissipation
in the boundary
layer.

Figure 3. Collage of contours of the time-mean streamfunction for different values of Re_i and Re_b . The contour interval is 0.2 in units where 1 is the maximum of the Sverdrup solution. Regions of negative streamfunction are shaded.