GFD 2007
Boundary Layers: Sloping bottoms in a stratified, rotating fluid

In oceanic coastal regions, e.g. on the shelf regions between the coast and the deep ocean, the bottom generally slopes and the fluid is stratified.

We have already seen the way the thermal boundary layers on vertical walls can control the interior flow and how the Ekman layers on horizontal boundaries can do the same for rotating fluids. Sloping boundaries are a type of hybrid of these two.

The study of the boundary layer on the sloping bottom boundary introduces some new and fascinating features to the analysis. We can only touch on some of the issues in these lectures.

References

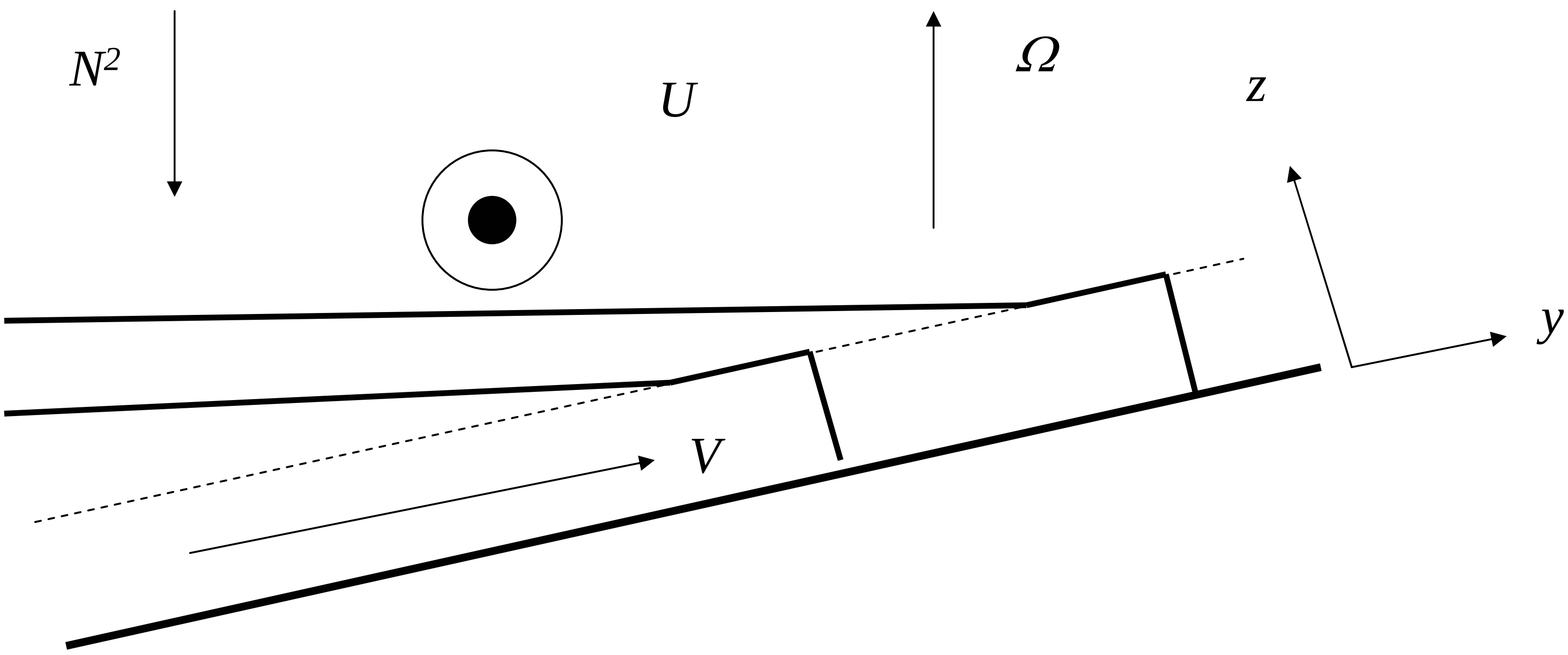
Chapman, D. and S. Lentz. 1997 Adjustment of stratified flow over a sloping bottom. *J. Phys. Ocean.*,**27**, 340-356.

Garrett, C. , P. MacCready, and P.B. Rhines 1993 Boundary mixing and arrested Ekman layers: Rotating stratified flow near a sloping boundary, *Annu.Rev. Fluid Mech.* **25**, 291-323

MacCready, P. and P.B. Rhines 1993 Slippery bottom boundary layers on a slope. *J.Phys. Ocean.* **23**, 5-22.

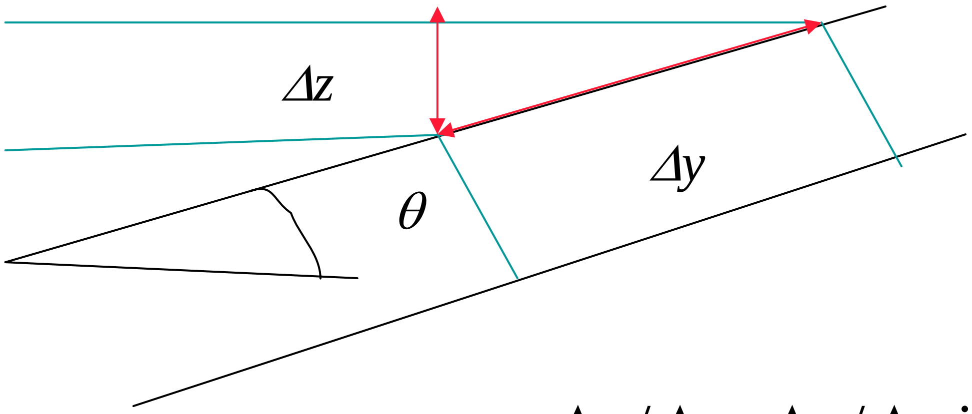
Trowbridge, J.H. and S. Lentz. 1991 Asymmetric behavior of an oceanic boundary layer above a sloping bottom. *J. Phys. Ocean.* ,**21**, 1171-1185

A schematic of the bottom boundary layer: upwelling case



Flow in x direction *U*.Ekman flux up the slope from high to low pressure.

The along slope density gradient



$$\Delta z = \Delta y \sin\theta$$

In boundary layer

$$\Delta\rho / \Delta z = \Delta\rho / \Delta y \sin\theta$$

$$\Rightarrow g\frac{\partial\rho}{\partial y} = -\rho N^2 \sin\theta, \qquad N^2 = -\frac{g}{\rho}\frac{\partial\rho}{\partial z}$$

Thermal wind in boundary layer

Fluid moving up the slope a distance will produce a density anomaly

$$\Delta\rho = \frac{1}{g} N^2 \rho_o \sin\theta \Delta y$$

$$f = 2\Omega\cos\theta \sim 2\Omega$$

$$f \frac{\partial U}{\partial z} = \frac{g}{\rho}\frac{\Delta\rho}{\Delta y} = N^2 \sin\theta$$

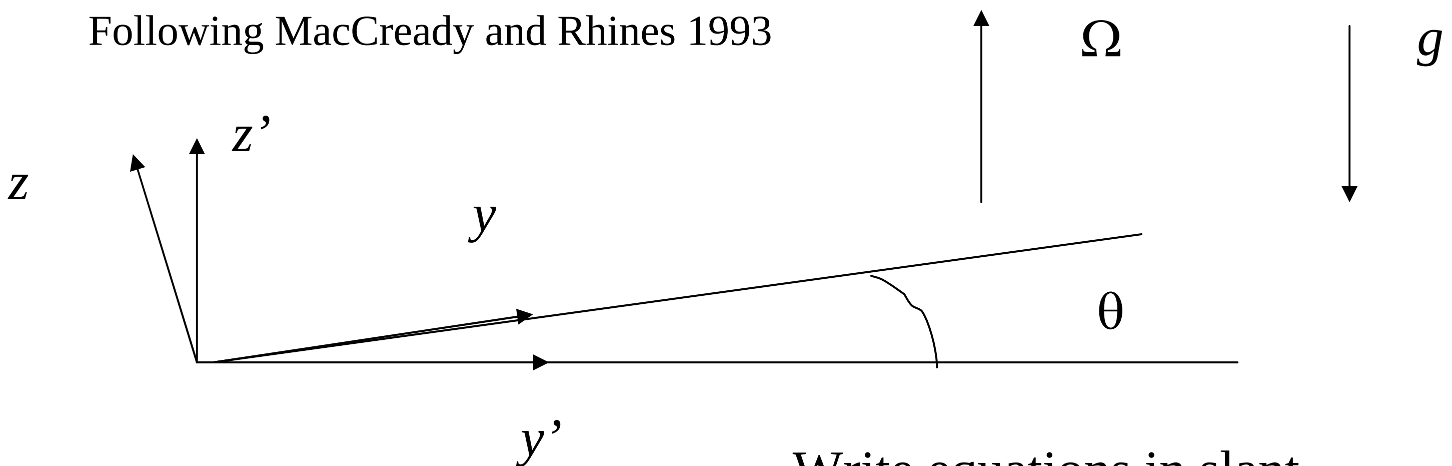
and over a depth of the bottom boundary layer of the order

$$H = \frac{fU}{N^2 \sin\theta}$$

it would be possible to adjust the speed of the current to zero without Ekman layers and their dissipation. Currents could flow long distances without decay.

Steady, laminar boundary layer on a sloping surface

Following MacCready and Rhines 1993



Write equations in slant coordinate frame.

$$v = v'\cos\theta + w'\sin\theta$$

$$w = -v'\sin\theta + w'\cos\theta$$

Far from the lower boundary

$$T_\infty = \Delta T_v z'/D$$

$$N = \left(g\alpha\Delta T_v / D\right)^{1/2}$$

Equations of motion (1)

$$T = \Delta T_v \frac{z'}{D} + \vartheta(y,z)$$
$$= \Delta T_v \left(z\cos\theta + y\sin\theta\right)/D + \vartheta(y,z)$$

where $\vartheta \longrightarrow$ 0 for large z

Search for solutions independent of y and are only functions of z.

w identically zero. For large z, $u \longrightarrow U$, a constant

v is independent of y

Nonlinear terms in momentum equation vanish identically.

Equations of motion (2)

$$2\Omega\cos\theta\; u = -\frac{1}{\rho_o}\frac{\partial \tilde{p}}{\partial y} + A v_{zz} + b\sin\theta$$

$$-2\Omega\cos\theta\; v = A u_{zz}$$

$$2\Omega\sin\theta u = -\frac{1}{\rho_o}\frac{\partial \tilde{p}}{\partial z} + b\cos\theta$$

$$v N^2 \sin\theta = \kappa b_{zz}$$

$$b = \alpha g \vartheta$$

$$f = 2\Omega\cos\theta$$

A is momentum mixing coefficient, κ is the thermal diffusivity.

$\tilde{p}$ is the pressure perturbation.

α is the coefficient of thermal expansion. u, v, b are independent of y.

$$\partial^2 \tilde{p}/\partial y \partial z = 0$$

Boundary conditions and boundary layer equation

$$u = 0, v = 0, b_z = -N^2 \cos\theta$$

Insulating condition on z =0

$$\frac{f^2}{A} v = A v_{zzzz} + b_{zz} \sin\theta$$

If the fluid were homogeneous or if the bottom were flat we would recover the Ekman layer problem. Using the thermal equation to eliminate *b* in favor of v yields

$$v_{zzzz} + 4 q^4 v = 0,$$

$$q^4 = \frac{1}{4}\left[\frac{f^2}{A^2} + \frac{N^2}{A\kappa}\sin^2\theta\right]$$

Solution for laminar boundary layer (1)

$$v = Ce^{-qz}\cos qz + Be^{-qz}\sin qz$$

$$q^4 = \frac{1}{4}\left[\frac{f^2}{A^2} + \frac{N^2}{A\kappa}\sin^2\theta\right]$$

If the bottom is flat the bl scale is the Ekman thickness. If the bottom is vertical, i.e. if θ=π/2 the scale is the buoyancy layer thickness

v =0 at z =0 means C =0. From the thermal equation

$$b_z = -B\frac{N^2\sin\theta}{2q\kappa}e^{-qz}\left[\cos qz + \sin qz\right]$$

And from $\longrightarrow$ $-2\Omega\cos\theta\ v = Au_{zz}$

$$\boxed{u = -\frac{f}{2Aq^2}Be^{-qz}\cos qz + U_\infty}$$

Solution for boundary layer (2)

Insulating condition $b_z = -N^2 \cos\theta$ $\longrightarrow$ $B = 2q\kappa \cot\theta$

While the no slip condition on *u* yields

$$\boxed{U_\infty = \frac{f}{Aq}\kappa \cot\theta}$$

$$u = U_\infty\left[1 - e^{-qz}\cos qz\right]$$

Flow at infinity is not arbitrary. It is part of the solution!

Thermal balance

Frictionally driven flow up the slope

$$\psi(z) = \int_0^z v\, dz = \kappa \cot\theta \left[1 - e^{-qz}\left(\cos qz + \sin qz\right)\right]$$

The total is $$\psi(\infty) = \kappa \cot\theta = \frac{Aq}{f} U_\infty$$

And this follows directly from the integral of the thermal equation

$$v N^2 \sin\theta = \kappa b_{zz} \qquad b_z = -N^2 \cos\theta, \quad z = 0$$

The control of the interior

This result, while at first glance non intuitive, is really just a manifestation of the control mechanisms we have already met in our discussion of the linear flow in the cylinder, although here in a more extreme form.

Part of our unease, is related to the sense that we ought to be able to drive the system as we would like and establish some equilibrium velocity along the isobaths that will differ from (3.2.14) or that we should, at least initially specify a different far field along shore flow.

We continue our presentation, following the work of MacCready, Rhines and Garrett by first considering the latter possibility.

The “slow” diffusion equation (1).

Same equations, w =0, but with time dependence.

$$\frac{\partial v}{\partial t} + 2\Omega\cos\theta\; u = -\frac{1}{\rho_o}\frac{\partial \tilde{p}}{\partial y} + Av_{zz} + b\sin\theta$$

$$\frac{\partial u}{\partial t} - 2\Omega\cos\theta\; v = Au_{zz}$$

$$2\Omega\sin\theta u = -\frac{1}{\rho_o}\frac{\partial \tilde{p}}{\partial z} + b\cos\theta$$

$$\frac{\partial b}{\partial t} + vN^2\sin\theta = \kappa b_{zz}$$

Scaling:

$$(u,v) = U(u',v'), \quad y = Ly', \quad z = Dz', \; t = \frac{D^2}{\kappa}t'$$

$$\tilde{p} = \rho_o fUL\tilde{p}', \quad b = \frac{fUL}{D}b'$$

Non dimensional equations

$\delta = D/L$

$$\frac{E}{2\sigma}\frac{\partial v}{\partial t} + u = -\frac{\partial \tilde{p}}{\partial y} + \frac{E}{2} v_{zz} + b\left(\frac{\sin\theta}{\delta}\right),$$

$$\frac{E}{2\sigma}\frac{\partial u}{\partial t} - v = \frac{E}{2} u_{zz}$$

$$-\tan\theta\ \delta u = -\frac{\partial \tilde{p}}{\partial z} + b\cos\theta$$

$$\frac{E}{2\sigma}\frac{\partial b}{\partial t} + \left(\frac{\sin\theta}{\delta}\right) S v = \frac{E}{2\sigma} b_{zz}$$

$$f = 2\Omega\cos\theta \qquad E = {2A}\big/{fD^2}, \quad S = \frac{N^2\delta^2}{f^2}$$

Interior equations

$$u_I = -\frac{\partial p_I}{\partial y} + b_I \left(\frac{\sin\theta}{\delta} \right)$$

$$b_I \cos\theta = \frac{\partial p_I}{\partial z} - \delta u_I \tan\theta$$

Eliminating *v* between

$$\frac{E}{2\sigma}\frac{\partial u}{\partial t} - v = \frac{E}{2} u_{zz}$$

$$\frac{E}{2\sigma}\frac{\partial b}{\partial t} + \left(\frac{\sin\theta}{\delta} \right) S v = \frac{E}{2\sigma} b_{zz}$$

yields

$$\frac{1}{\sigma}\frac{\partial}{\partial t}\left[u_I - \frac{\delta}{S\sin\theta} b_I \right] = \frac{1}{2}\frac{\partial^2}{\partial z^2}\left[u_I - \frac{\delta}{\sigma S\sin\theta} b_I \right]$$

Eliminating the pressure in the first two equations

and noting that *b* is independent of *y*

$$\boxed{\frac{\partial u_I}{\partial z} = \frac{\partial b_I}{\partial z}\left[\frac{\sin\theta}{\delta} \right]}$$

The slow diffusion equation (2)

$$\frac{\partial}{\partial t}\left(\frac{\partial u_I}{\partial z}\right) = \sigma \frac{\left\{1+\dfrac{\delta^2}{\sigma S \sin^2\theta}\right\}}{\left\{1+\dfrac{\delta^2}{S \sin^2\theta}\right\}} \frac{\partial^2}{\partial z^2}\left(\frac{\partial u_I}{\partial z}\right)$$

$$S\frac{\sin^2\theta}{\delta^2} = \frac{N^2}{f^2}\sin^2\theta \equiv S_*$$

$$\mu_{diff} = \sigma\left(\frac{\dfrac{1}{\sigma}+S_*}{1+S_*}\right)$$

Effective diffusion coefficient

$$\left(\mu_{diff}\right)_{dimensional} = A\left(\frac{\dfrac{1}{\sigma}+S_*}{1+S_*}\right)$$

dimensional

if $\sigma>1$, the diffusion coefficient would be smaller than in the absence of stratification

Final form of slow diffusion eqn.

$$\frac{\partial u_I}{\partial t} = \sigma \frac{\left\{1 + \dfrac{\delta^2}{\sigma S \sin^2\theta}\right\}}{\left\{1 + \dfrac{\delta^2}{S \sin^2\theta}\right\}} \frac{\partial^2 u_I}{\partial z^2}$$

If u is independent of z and t at z--> *inf*.

To obtain boundary conditions for s.d.e need to consider boundary layer at sloping bottom.

Boundary layer coordinate

$$\zeta = z E^{-1/2}$$

Label correction variables with *e*

Boundary layer equations for correction functions

$$u_e = -\frac{\partial p_e}{\partial y} + \frac{1}{2} v_{e\zeta\zeta} + b_e \frac{\sin\theta}{\delta},$$

$$-v_e = \frac{1}{2} u_{e\zeta\zeta},$$

$$\delta \tan\theta_e = -\frac{1}{E^{1/2}} \frac{\partial p_e}{\partial \zeta} + b_e \cos\theta,$$

$$\frac{\sin\theta}{\delta} v_e = \frac{1}{2\sigma S} b_{e\zeta\zeta}$$

These are the same steady equations we dealt with before and they yield

$$\frac{\partial^4 v_e}{\partial z^4} + 4\left[1 + \sigma S \frac{\sin^2\theta}{\delta^2}\right] v_e = 0.$$

Boundary layer solution

$$v_e = Ae^{-\alpha\zeta}\cos\alpha\zeta + Be^{-\alpha\zeta}\sin\alpha\zeta,$$

$$b_e = \sigma S\frac{\sin\theta}{\delta}\left[e^{-\alpha\zeta}\left\{(A-B)\sin\alpha\zeta - (A+B)\cos\alpha\zeta\right\}\right],$$

$$u_e = \frac{1}{\alpha^2}\left[Ae^{-\alpha\zeta}\sin\alpha\zeta - Be^{-\alpha\zeta}\cos\alpha\zeta\right],$$

$$\alpha = \left[1 + \sigma S\frac{\sin^2\theta}{\delta^2}\right]^{1/4}$$

Matching conditions

$$u_I + u_e = 0,$$

$$v_I + v_e = 0,$$ v_I is O(E) $\rightarrow$ B =0

$$\frac{\partial b_I}{\partial z} + E^{-1/2}\frac{\partial b_e}{\partial \zeta} = -\frac{S}{\varepsilon}\cos\theta,$$ $\rightarrow$ A=0

$$\varepsilon = U / fL$$

The frictional boundary layer vanishes to lowest order.

u_I must satisfy the no-slip bc at z =0

Diffusion solution

$$u_I = U_\infty \frac{2}{\sqrt{\pi}} \int_0^{\zeta/(\mu_{dif} t)^{1/2}} e^{-\varphi^2} d\varphi$$

Next order boundary layer solution still has v_e =0--> A=0

Boundary layer contribution to buoyancy flux yields

$$B = -\frac{\delta\alpha}{2\sigma S \sin\theta}\left[\frac{S}{\varepsilon}\cos\theta + \frac{\delta U_\infty}{\sin\theta\sqrt{\pi\mu_{diff} t}}\right]$$

Long time solution in boundary layer

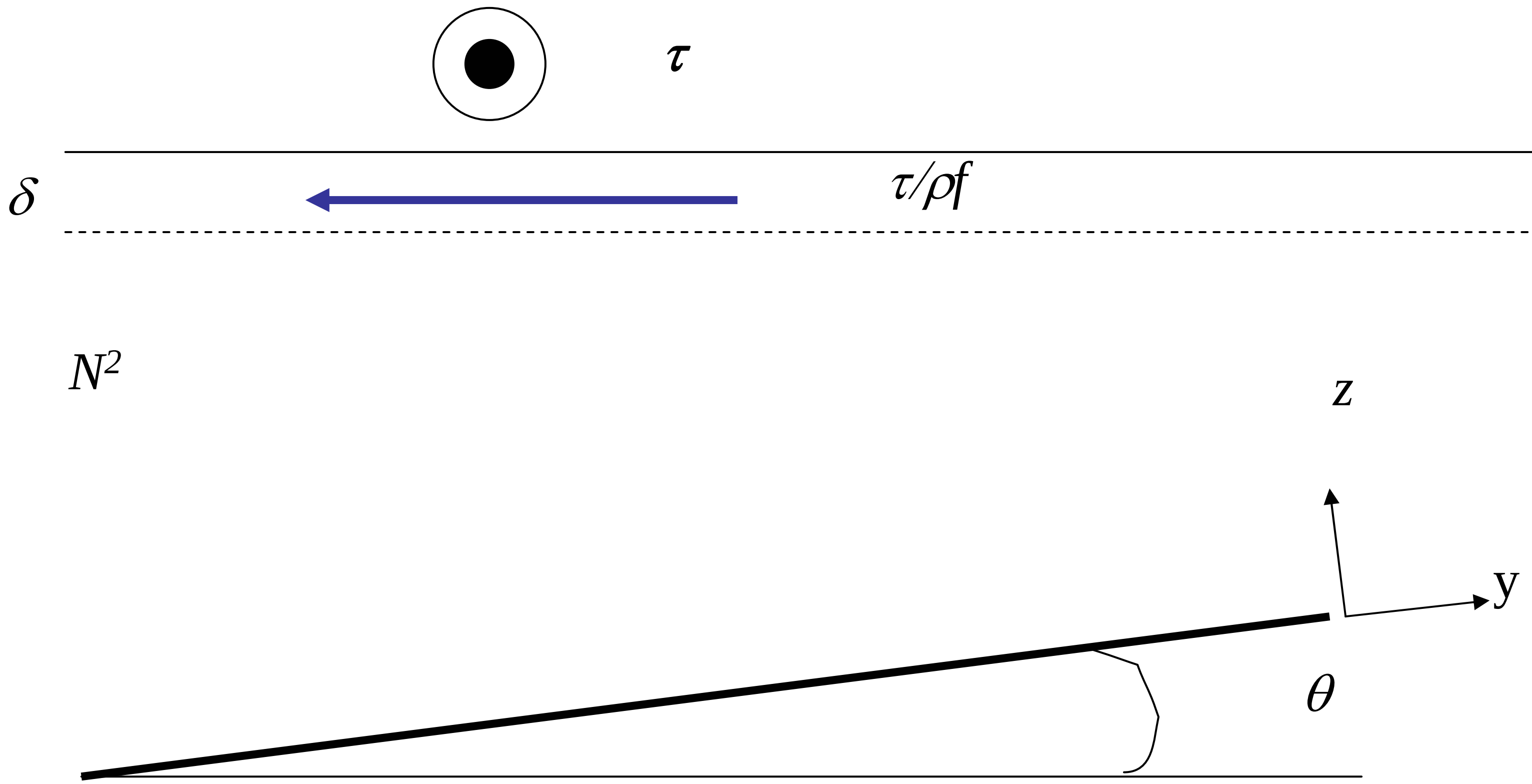
As t goes to infinity

$$b_e = -E^{1/2}\frac{S}{\varepsilon\alpha}\cos\theta e^{-\alpha\zeta}\cos\alpha\zeta$$

which is the steady state solution (in n.d. form) already attained.

Hence it is possible to consider arbitrary interior flows but, at least with the simple physics here, the boundary layer control eventually expunges the along isobath flow and yields an asymptotically weak frictional boundary layer. This, in one sense resolves the conundrum posed by the steady boundary layer solution in which the interior flow and the cross shelf flow depended only on the stratification and the vertical thermal diffusion coefficient. Nevertheless, the solution presented here eventually approaches that very constrained solution.

Richer Physics
Stress driven flow



Off shore Ekman flux in upper Ekman layer

Equations of motion

variations in the wind stress with y will force an Ekman pumping at the base of the Ekman layer $w_e(y)$. Assume θ is small.

In slant frame

$$fu = -\frac{1}{\rho_o} p_y + b\sin\theta + A_v v_{zz} + A_H v_{yy},$$

$$-fv + fw\tan\theta = A_v u_{zz} + A_H u_{yy},$$

$$-f\tan\theta u = -\frac{1}{\rho_o} p_z + b\cos\theta$$

$$v_y + w_z = 0,$$

$$v\left[N^2\sin\theta + b_y\right] + w\left[N^2\cos\theta + b_z\right] = \kappa_V b_{zz} + \kappa_H b_{yy}$$

At base of Ekman layer

$$w = w_e = -\frac{1}{\rho_o f}\frac{\partial\tau}{\partial y}$$

Interior solution (1)

Ignore friction in momentum equations. v is O(E)=A_V/fD^2. Implies w is independent of z and hence equal w_e everywhere in the interior. The thermal equation becomes.(we have assumed an interior solution for b independent of z)

$$wN^2 = \kappa_H b_{yy}$$

$$-\frac{1}{\rho_o f}\frac{\partial \tau}{\partial y}N^2 = \kappa_H b_{yy}$$

$$\longrightarrow \quad b_{I_y} = -\frac{1}{\rho_o f}\frac{N^2}{\kappa_H}\tau(y)$$

Assuming b_I vanishes at large negative y where the stress vanishes

Interior solution (2)

$$u_I = \frac{N^2}{\kappa_H}\frac{\tau}{\rho_o f^2} z + u_o(y)$$

From thermal wind eqn.

Unknown barotropic contribution

The total interior buoyancy diffusive flux perpendicular to the lower boundary is:

$$\Im_z = N^2\left[\kappa_v \cos\theta + \frac{\tau}{\rho_o f_o}\sin\theta\right]$$

Bottom boundary layer (again)

$$v_{b_{zzzz}} + \frac{4}{l^{4}} v_b = 0,$$

$$l^{-4} = \frac{f^{2}}{4A_v^{2}}\left[1 + \frac{N^{2}\sin^{2}\theta}{f^{2}\kappa_v / A_v}\right]$$

$$v_b = e^{-z/l}\left[A\cos z/l + B\sin z/l\right]$$

$$u_b = \frac{f}{A_v}\frac{l^{2}}{2}e^{-z/l}\left[A\sin z/l - B\cos z/l\right]$$

$$\frac{\partial b_b}{\partial z} = -\frac{N^{2}}{2\kappa_v}\delta\sin\theta e^{-z/l}\left[A\left(\cos z/l - \sin z/l\right) + B\left(\cos z/l + \sin z/l\right)\right],$$

$$w_b = \frac{l}{2}e^{-z/l}\left[A_y\left(\cos z/l - \sin z/l\right) + B_y\left(\cos z/l + \sin z/l\right)\right]$$

Matching

$$u_o - \frac{f}{A_v}\frac{l^2}{2}B = 0$$ For u

$$-\frac{1}{\rho_o f}\frac{\partial \tau}{\partial y}\sin\theta + A = 0,$$ For v

$$-\frac{1}{\rho_o f}\frac{\partial \tau}{\partial y} + \frac{l}{2}(A+B) = 0,$$ For w

redundant

$$\kappa_V N^2\cos\theta + \frac{N^2\tau\sin\theta}{\rho_o f} - N^2\frac{l}{2}\sin\theta(A+B)$$ For b_z

Solution

$$\frac{\ell}{2}(A+B)=\frac{\tau}{\rho_o f}+\kappa_v \cot\theta$$

Boundary layer mass flux

Off shore flux balancing on shore Ekman flux

Transport in bottom boundary layer induced by stratification and diffusion as in earlier solution

finally

$$A=\frac{1}{\rho_o f}\frac{\partial \tau}{\partial y}\sin\theta,$$

$$B=-\frac{1}{\rho_o f}\frac{\partial \tau}{\partial y}\sin\theta+\frac{2}{\ell}\frac{\tau}{\rho_o f}+\frac{2}{\ell}\kappa_v \cot\theta$$

$$u_o=-\frac{f}{A_v}\frac{\ell^2}{2}B$$

Definitions

"vertical Ekman #

$$E_v = \frac{2A_v}{fD^2}, \quad \sigma_v = \frac{A_v}{\kappa_v}$$

"horizontal" Ekman#

$$E_H = \frac{2A_H}{fL^2}, \; \sigma_H = \frac{A_H}{\kappa_H}$$

"vertical" and "horizontal" Prandlt numbers

Interior long shore velocity

$$u_I = \underbrace{\frac{N^2}{\kappa_H f}\frac{\tau}{\rho_o f}}_{term\ 1} z + \frac{f}{A_v}\frac{\lambda^2}{2}\left[\underbrace{\frac{2}{\lambda}\frac{\tau}{\rho_o f}}_{term\ 2} - \frac{1}{\rho_o f}\frac{\partial \tau}{\partial y}\sin\theta + 2\kappa_v \cot\theta\right]$$

Ratio of terms

$$\frac{f}{A_v}\frac{\lambda^2}{2} \approx 1$$

$$\frac{\text{term 1}}{\text{term 2}} = \frac{N^2}{f^2}\frac{f}{\kappa_H} D\lambda$$

$$= \frac{N^2 D^2}{f^2 L^2}\frac{f L^2}{A_H(\kappa_H / A_H)}\frac{\lambda}{D}$$

$$= \frac{\sigma_H S}{E_H} E_v^{1/2} = \left[\frac{\sigma_H S}{E_v^{1/2}}\right]\left(\frac{E_v}{E_H}\right)$$

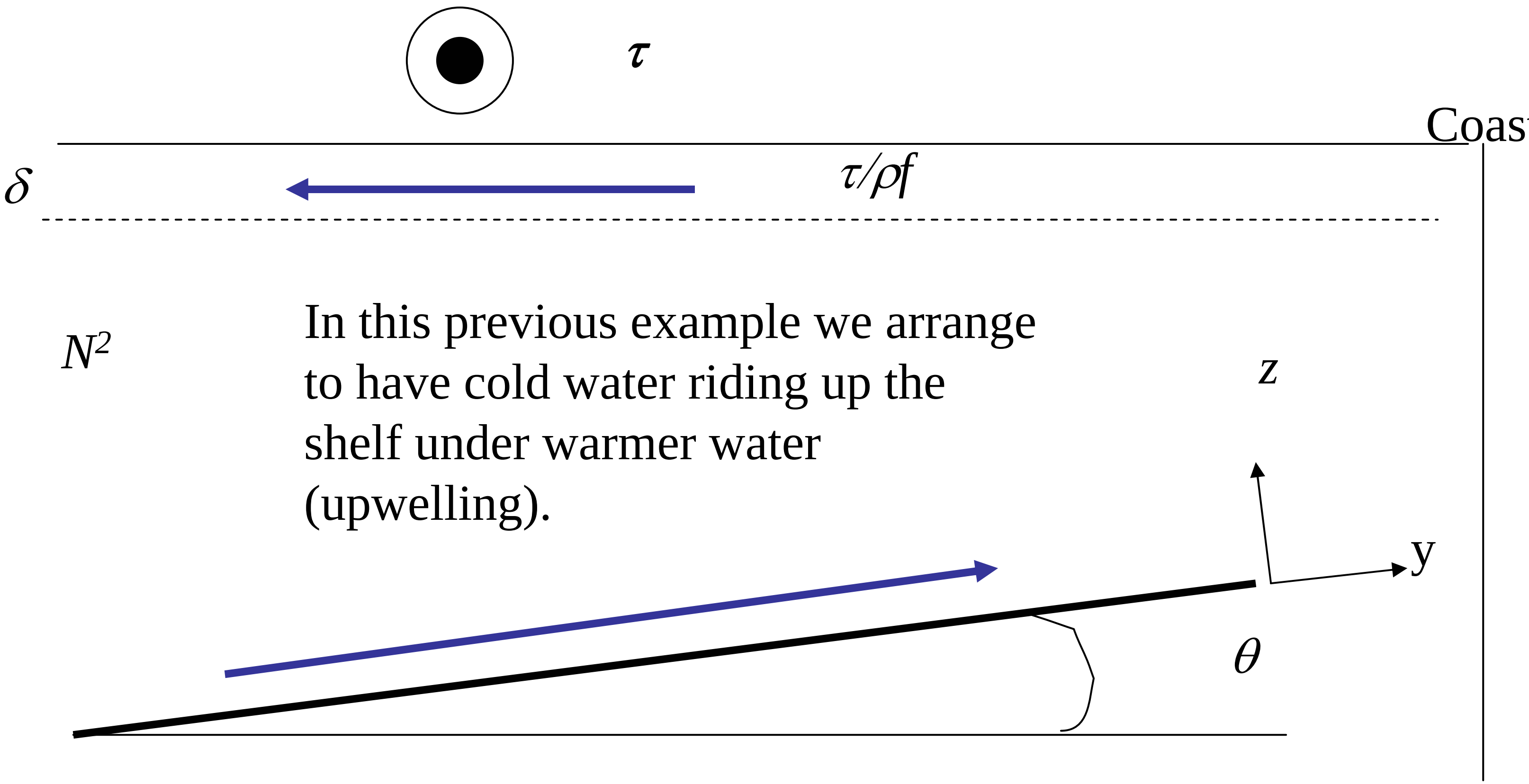
If E_V/E_H= O(1) then if

$$\boxed{\sigma_H S >> E_v^{1/2}}$$

Then the interior velocity very nearly satisfies the no slip condition on z=0 without the b.l.

As before

The turbulent bottom boundary layer



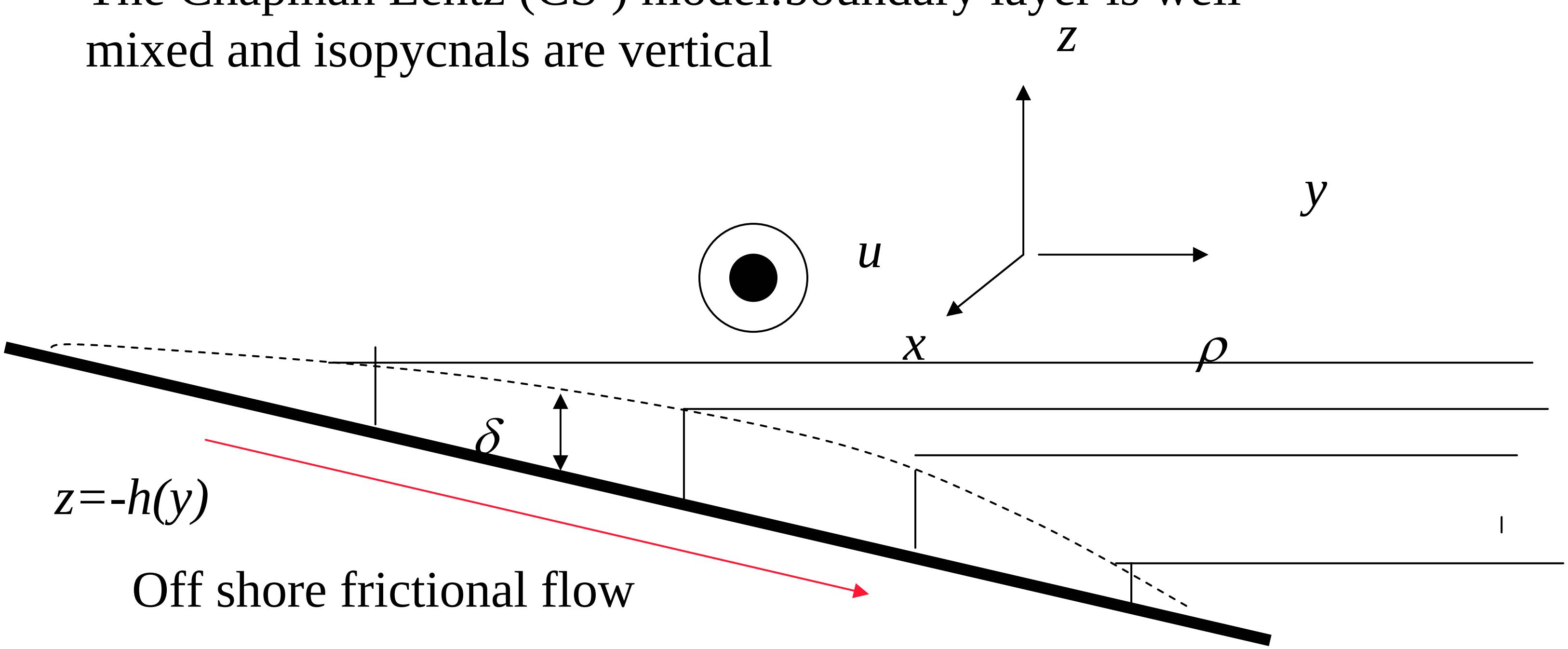
In this previous example we arrange to have cold water riding up the shelf under warmer water (upwelling).

This enhances the static stability and a laminar model is at least plausible.The situation is different in downwelling

Chapman Lentz Model

Light water driven under denser water. Expect convective mixing and a thick turbulent boundary layer. See Trowbridge and Lentz (1991) and Chapman and Lentz (1997).

The Chapman Lentz (CS) model:boundary layer is well mixed and isopycnals are vertical



CL model:
equations of motion

$$
\begin{aligned}
-fv &= -p_x/\rho_o + \tau^x{}_z/\rho_o \\
fu &= -p_y/\rho_o \\
0 &= -p_z - \rho g \\
0 &= u_x + v_y + w_z \\
u\rho_x + v\rho_y + w\rho_z &= B_z
\end{aligned}
$$

Linear momentum eqn.

Non linear thermal eqn.

τ^x is stress in fluid in x direction.

B is the vertical turbulent density flux.

At the bottom, $z=-h_b(y)$

$B_z = 0$ and

$$\tau^x(-h) = -ru_b$$

The CL adjustment problem

CL examine the evolution of a coastal current.

It starts as a narrow current upstream and spreads laterally due to friction.

The boundary layer thickens as the current flows downstream.

The current outside the boundary layer is not sheared vertically.

Using integral budgets for mass, momentum and buoyancy, eqns for the boundary layer thickness δ are found as well as the interior u.

Results of CL problem (1)

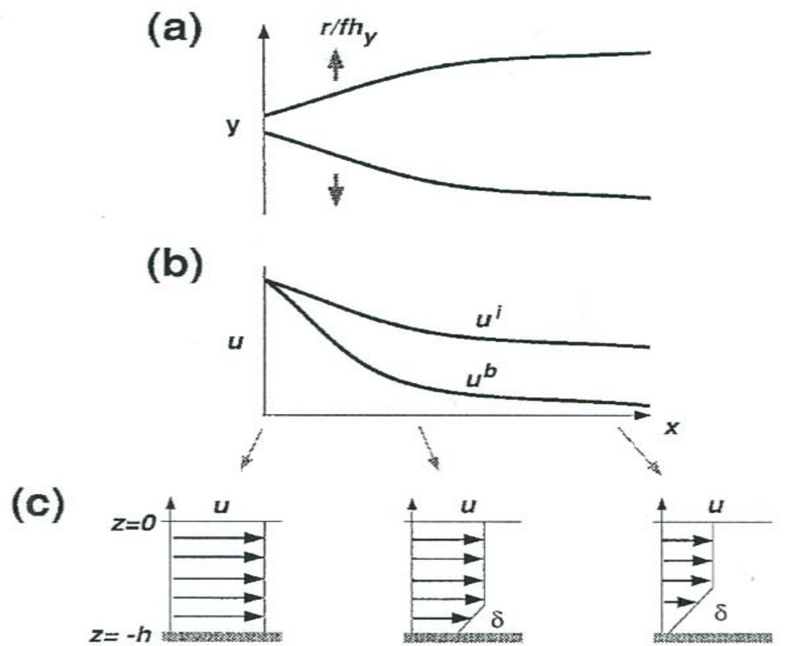


FIG. 2. Schematic depicting the adjustment and evolution of a narrow inflow starting at $x = 0$. (a) Plan view of the current boundaries that initially spread, owing to bottom friction, at a rate set by r/fh_y. (b) Evolution of the interior velocity u^i and bottom velocity u^b with downstream distance. (c) Along-isobath velocity profiles at various stages downstream. The bottom boundary layer grows, while the interior and bottom velocities both decrease, eventually reaching an equilibrium where the bottom velocity vanishes.

The boundary layer evolves until finally the thermal wind brings *u* to zero and eliminates the bottom stress. The scaling for the bl thickness is, as before,

$$H = \frac{fU}{N^2 \sin\theta}$$

Results of CL problem (2)

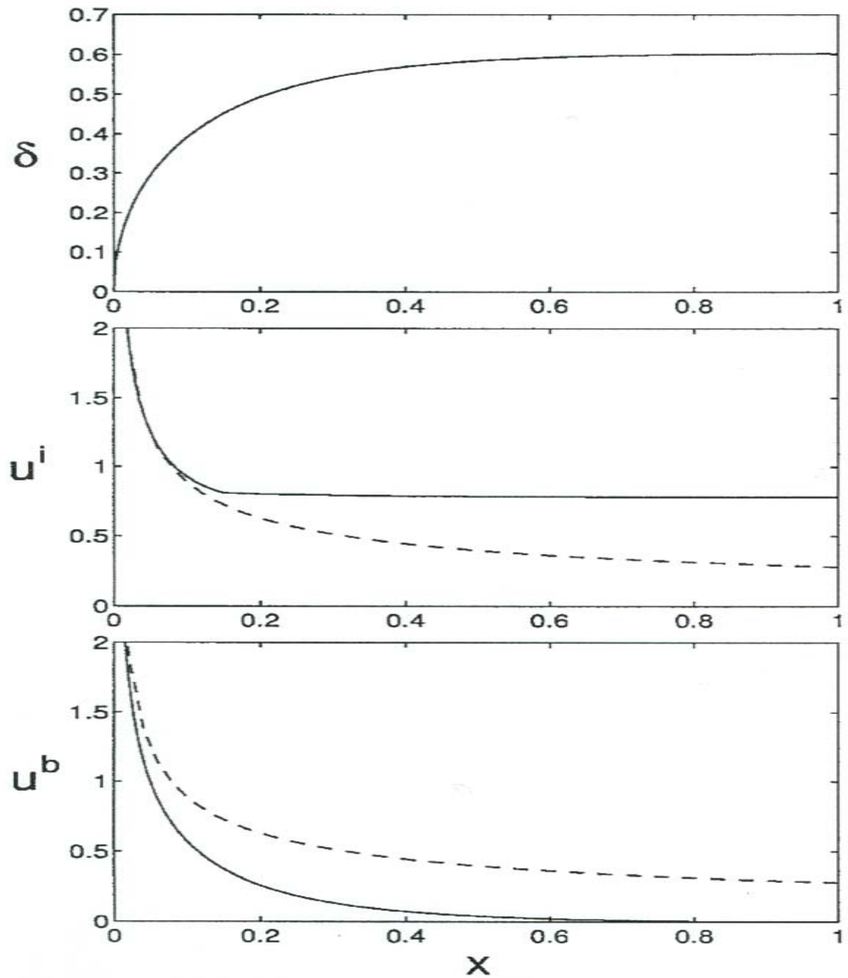


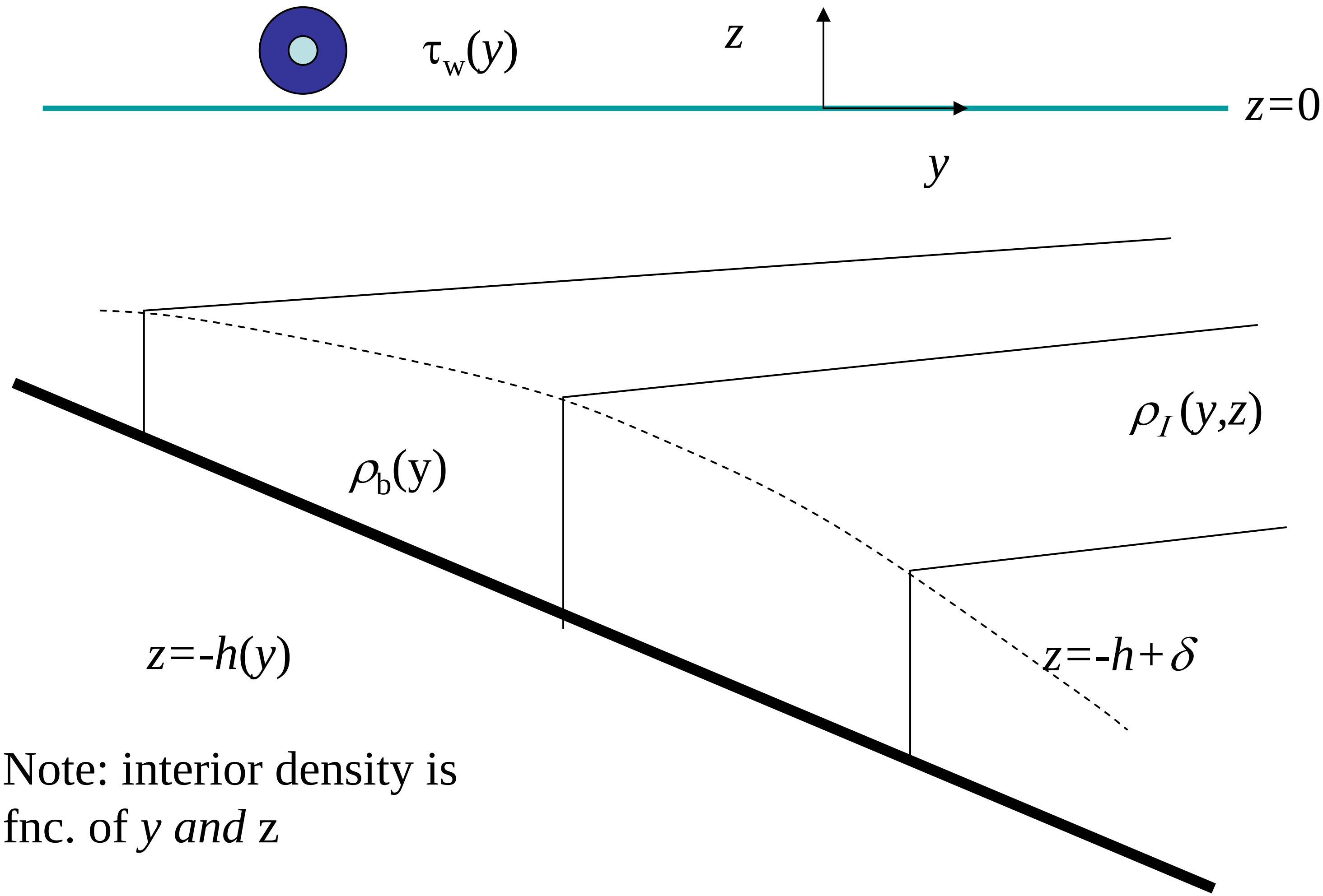
FIG. 5. Maximum values of (upper) bottom boundary layer thickness, (middle) interior along-isobath velocity, and (lower) bottom velocity at each downstream (x) location for the stratified flow shown in Fig. 3. Dashed curves correspond to the unstratified flow in Fig. 4.

Dashed curves for N=0.

Note the stratified current flows without further decay after u_b goes to zero.

Want a model to discuss this equilibrium state that does not require a 3-d numerical calculation

Stress driven turbulent bottom boundary layer: The CL model



Note: interior density is fnc. of *y and* z

Equations of motion

Motion is assumed two dimensional--- independent of *x*

Boundary layer must carry off shore flow.

$$fu = -\frac{1}{\rho_o}\frac{\partial p}{\partial y},$$

$$-fv = \frac{1}{\rho_o}\frac{\partial \tau}{\partial z},$$

τ is turbulent stress *in fluid*

$$\frac{\rho}{\rho_o} g = -\frac{1}{\rho_o}\frac{\partial p}{\partial z},$$

ρ density anomaly

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 .$$

$$v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial}{\partial y}\left(\kappa_H \frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_V \frac{\partial \rho}{\partial z}\right)$$

Interior(1)

Onshore flux in upper Ekman layer $M_E = \tau^w / \rho_o f$

Ekman pumping velocity

$$w = w_e = -\frac{1}{\rho_o f}\frac{\partial \tau^w}{\partial y}$$

Buoyancy flux normal to bottom vanishes and all perturbations go to zero for large y. The density is continuous at

$$z = -\alpha y + \delta$$

Below the Ekman layer and above the bbl the turbulent stress in the fluid interior is zero. Thus

$$v_I = 0 \qquad w_I = w_e(y) = -\frac{1}{\rho f}\frac{\partial \tau^w}{\partial y}$$

Interior (2)

$$w_e \frac{\partial \rho_I}{\partial z} = \frac{\partial}{\partial y}\left(\kappa_{H_I} \frac{\partial \rho_I}{\partial y} \right) + \frac{\partial}{\partial z}\left(\kappa_{V_I} \frac{\partial \rho_I}{\partial z} \right)$$

$$\kappa_{V_I} \frac{\partial \rho_I}{\partial z} = -H \qquad z=0$$

let

$$\rho_I = \rho_o + \bar{\rho}_I(z) + \tilde{\rho}_I(y)$$

$$\frac{1}{\rho_o} \frac{\partial \bar{\rho}_I}{\partial z} = -\frac{H}{\kappa_{V_I} \rho_o} \equiv -\frac{1}{g} N^2$$

Interior density equation

$$-w_e\rho_o \frac{N^2}{g} = \frac{\partial}{\partial y}\left(\kappa_{H_I}\frac{\partial \tilde{\rho}_I}{\partial y}\right)$$

$$\longrightarrow \quad \frac{\rho_I}{\rho_o} = -z\frac{N^2}{g} - \int_y^{\infty} \frac{\tau^w}{\rho_o f}\frac{N^2}{g\kappa_{H_I}}\,dy' \quad +1$$

$$\frac{\partial u_I}{\partial z} = \frac{g}{f\rho_o}\frac{\partial \tilde{\rho}_I}{\partial y} = \frac{\tau^w N^2}{\rho_o f^2 \kappa_{H_I}} \quad \longrightarrow \quad u_I = \frac{z}{\kappa_{H_I}}\frac{N^2}{f^2}\frac{\tau^w}{\rho_o} - \frac{1}{\rho_o f}\frac{\partial p_s}{\partial y}$$

unknown

In the bottom boundary layer

Mixing is so intense density is only a function of y but is continuous with interior. Thus

$$\rho_b(y) = \rho_I(y, z = -h(y) + \delta)$$

$$\frac{\rho_b}{\rho_o} = (h-\delta)\frac{N^2}{g} - \int_y^\infty \frac{\tau^w}{\rho_o f}\frac{N^2}{g\kappa_{H_I}}\, dy' + 1$$

Pressure is hydrostatic and continuous with interior:

$$\frac{p_b}{\rho_o} = -(h-\delta)\left[z + (h-\delta)/2\right]N^2 + z\int_y^\infty \frac{\tau^w}{\rho_o f}\frac{N^2}{g\kappa_{H_I}}\, dy' - gz + \frac{p_s}{\rho_o}$$

$$u_b = \frac{N^2}{f}\frac{\partial}{\partial y}\left[\left(z + [h-\delta]/2\right)(h-\delta)\right] + z\frac{\tau^w}{\rho_o f^2}\frac{N^2}{\kappa_{H_I}} - \frac{1}{\rho_o f}\frac{\partial p_s}{\partial y}$$

Integral of x momentum eqn in bl

$$V_b\delta = \frac{\tau(z=-h)}{\rho_o f} \qquad\qquad V_b = \frac{1}{\delta}\int_{-h}^{-h+\delta} v_b\,dz$$

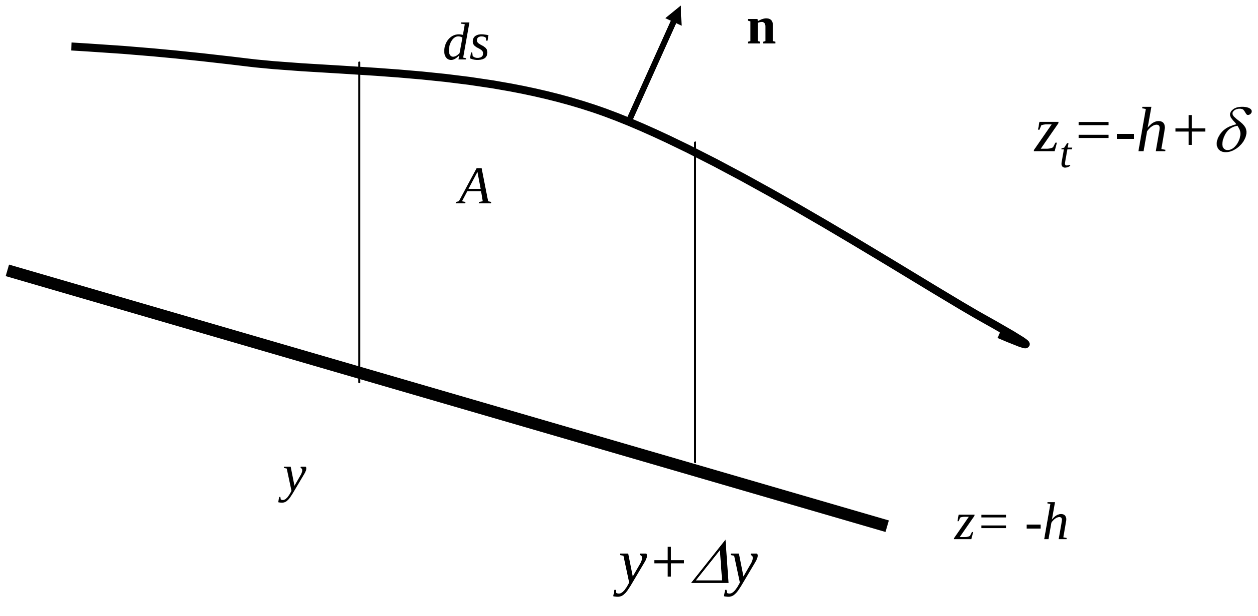
As in CL

$$V_b\delta = \frac{\tau(-h)}{\rho_o f} = r u_b(-h)/f$$

If we know V_b we then know the surface pressure gradient for a given boundary layer thickness (still unknown)

Mass budget for boundary layer

It is obvious that the off shore mass flux must balance the on shore Ekman flux. But it is illuminating to examine the detailed budget in preparation for the buoyancy budget.



Mass balance

$$\frac{\partial}{\partial y}\int_{z_t}^{z_b} v_b\,dz + w_b(z_t) - v_b(z_t)\frac{\partial z_t}{\partial y} - w_b(z_b) + v_b(z_b)\frac{\partial z_b}{\partial y} = 0$$

cancel

Flux across upper bdy of bbl

$$\hat{n} = \frac{\hat{k} - \hat{j}\dfrac{\partial z_t}{\partial y}}{\left(dy^2 + dz_t^{\ 2}\right)^{1/2}}\,dy$$

or

$$\frac{\partial V_b \delta}{\partial y} + w_* = 0$$

$$w_* = w_b(z_t) - v_b(z_t)\frac{\partial z_t}{\partial y}$$

Mass continuity with interior

Since $v_I = 0$ w_* must match w_e

$$\frac{\partial V_b \delta}{\partial y} + w_e = 0,$$

or

$$\frac{\partial V_b \delta}{\partial y} - \frac{\partial \tau^w / \rho_o f}{\partial y} = 0.$$

$$\longrightarrow \quad V_b \delta = \frac{\tau^w}{\rho_o f}$$

Buoyancy budget

Diffusive mass flux at upper bdy of bbl

$$dyV_b\delta\frac{\partial\rho_b}{\partial y}=\Im_b g\hat{n}\,ds+dy\frac{\partial}{\partial y}\left(\kappa_{H_b}\delta\frac{\partial\rho_b}{\partial y}\right)$$

$$V_b\delta\frac{\partial\rho_b}{\partial y}=\left[\kappa_{V_b}\frac{\partial\rho_b}{\partial z}-\kappa_{H_b}\frac{\partial\rho_b}{\partial y}\frac{\partial z_t}{\partial y}\right]_{z=z_t}+\frac{\partial}{\partial y}\left(\delta\kappa_{H_b}\frac{\partial\rho_b}{\partial y}\right)$$

Continuity of diffusive flux at top of bbl yields first term on rhs in terms of interior variables.

Final budget for buoyancy in bbl

$$V_b \delta \frac{\partial \rho_b}{\partial y} = \kappa_{V_I} \frac{\partial \rho_I}{\partial z} - \kappa_{H_I} \frac{\partial \rho_I}{\partial y} \frac{\partial z_t}{\partial y} + \frac{\partial}{\partial y}\left[\delta \kappa_{H_b} \frac{\partial \rho_b}{\partial y} \right]$$

After some algebra

$$\frac{\partial}{\partial y}\left[\kappa_{H_b} \delta \frac{\partial}{\partial y}(h-\delta) + \kappa_{H_b} \delta \frac{\tau^w}{\rho_o f \kappa_{H_I}} \right] = \kappa_{V_I} + \left(\frac{\tau^w}{\rho_o f} \right)^2 / \kappa_{H_I}$$

Eqn. for δ in terms of which

$$-\frac{1}{\rho_o f} \frac{\partial p_s}{\partial y} = \frac{N^2}{f} \delta \frac{\partial (h-\delta)}{\partial y} + h \frac{\tau^w N^2}{\rho_o f^2 \kappa_{H_I}} + \frac{\tau^w}{\rho_o r}$$

velocity in boundary layer, interior velocity

$$u_b = \frac{N^2}{f}(z+h)\left[\frac{\partial(h-\delta)}{\partial y} + \frac{\tau^w}{\rho_o f \kappa_{H_I}}\right] + \frac{\tau^w}{\rho_o r}$$

$$u_I = \frac{N^2}{f}(z+h)\left[\frac{\tau^w}{\rho_o f \kappa_{H_I}}\right] + \frac{N^2}{f}\delta\frac{\partial}{\partial y}(h-\delta) + \frac{\tau^w}{\rho_o r}$$

Continuous at $z=-h+\delta$

Note that in the absence of stress the bottom velocity in the bbl is zero as in MacCready and Rhines

ODE for δ (1)

Take bottom of form $h=-\alpha y$. Scale lengths with L and thickness of bbl with αL.

Consider a stress of form. $\tau^{w} = \tau_{o} e^{-a(y/L)}$

$$\frac{d}{dy}\left[\delta\frac{d\delta}{dy} - (1 + Fe^{-ay})\delta\right] = -\Sigma_V - F^2\Sigma_H e^{-2ay}$$

$$F = \frac{\tau_o}{\alpha\rho_o f\kappa_{H_I}},$$

$$\Sigma_V = \frac{\kappa_{V_I}}{\kappa_{H_b}\alpha^2}, \qquad \Sigma_H = \frac{\kappa_{H_I}}{\kappa_{H_b}}$$

ODE for δ (2)

$$\delta\frac{d\delta}{dy}-(1+Fe^{-ay})\delta=-\Sigma_V(y-y_o)-\frac{F^2}{2a}\left[e^{-2ay}-e^{-2ay_o}\right]\Sigma_H+C,$$

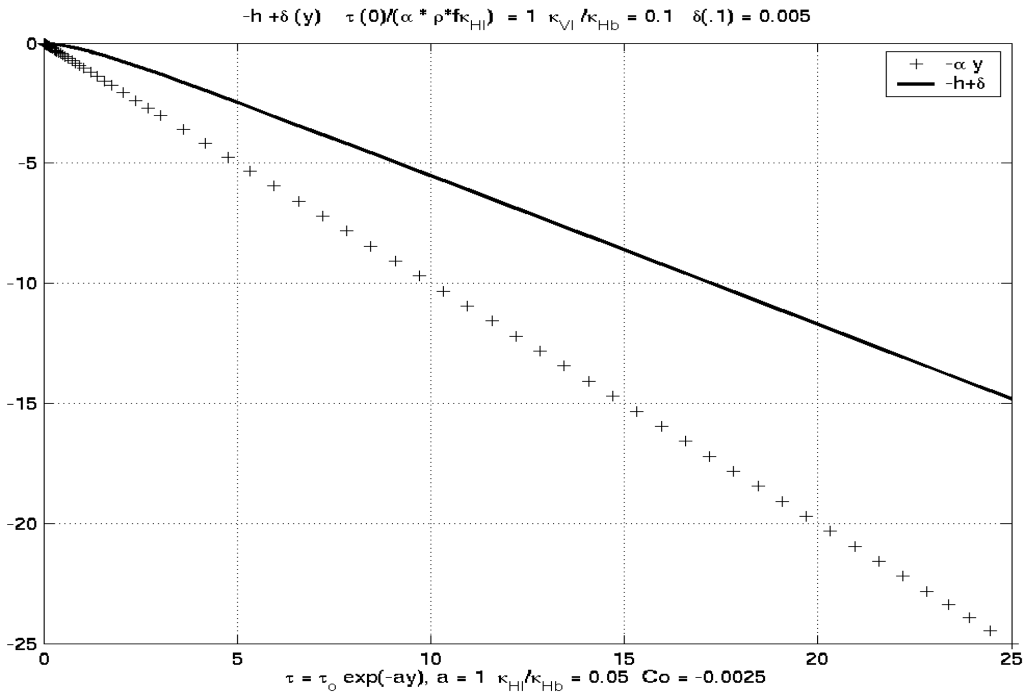
$$C=\left\{\delta\left[\frac{d\delta}{dy}-(1+Fe^{-ay_o})\right]\right\}_{y=y_o}$$

y_o is starting point for integration (model not valid in apex of wedge)

$$C=\left(\delta\left[-\frac{g}{\alpha N^2\rho_o}\frac{\partial\rho_b}{\partial y}\right]\right)_{y=y_o}$$

< 0

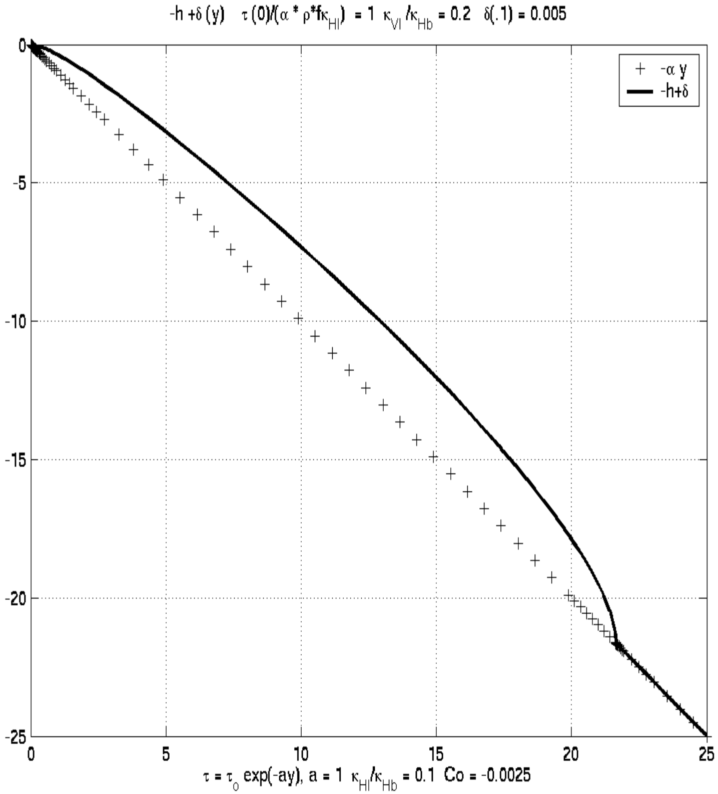
Results



The boundary layer thickness with respect to the sloping bottom for a =1, Σ_v = 0.1 , Σ_H= 0.05. A starting value of δ of half the depth at $y = y_o$ = 0.01 is chosen and C is -0.0025. F =1 has been used.

Results (2)

For larger diffusion coefficients in the interior



$$\Sigma_V = 0.2$$

$$\Sigma_H = 0.1$$

δ goes to zero well beyond the region of wind stress.

The interior velocity then satisfies the zero velocity condition

When δ goes to zero

$$u_I = \frac{N^2}{f}(z+h)\left[\frac{\tau^w}{\rho_o f \kappa_{H_I}}\right] + \frac{N^2}{f}\delta\frac{\partial}{\partial y}\left(h-\delta\right) + \frac{\tau^w}{\rho_o r}$$

Ratio of last term to first term then is:

$$\frac{N^2H^2}{f^2L^2}\frac{fL^2}{\kappa_{H_I}}\frac{r}{Hf} \approx \frac{\sigma S}{E_H}E_v^{1/2} = \frac{\sigma S}{E_v^{1/2}}\frac{E_v}{E_H}$$

So again, when

$$\frac{\sigma S}{E_v^{1/2}} >> 1$$ interior velocity satisfies bc.