Chapter 2.

Quasi-Geostrophic Theory: Formulation (review)

2.1 Introduction

For most of the course we will be concerned with instabilities that an be analyzed by the quasi-geostrophic equations. These are valid for synoptic scale motions in both the atmosphere and the ocean although in both cases the equations lose validity when the motion is very intense, e.g. for frontal instabilities. Even there, however, the quasi-geostrophic model often gives a useful qualitative picture of the instability. Quasi-geostrophic (q.g.) theory applies when the Rossby number is small and when several other side conditions are satisfied. A careful derivation of the q.g. equations is given in Chapter 6 of Geophysical Fluid Dynamics and for the purpose of the course I will just present in a heuristic fashion the main results. You should refer to the reference if the foundation of an assertion made below seems unclear. We shall also use the beta-plane approximation which allows us to simplify the algebra by using a locally Cartesian coordinate system.

We suppose that:

$$\varepsilon = U / f_o L << 1, \qquad f_o = 2\Omega \sin\theta_o$$

$$b = \beta L / f_o << 1, \qquad \beta = 2\Omega \cos\theta_o / R \qquad \text{(2.1.1 a,b)}$$

In the above, the characteristic scale for the horizontal velocity is U, the characteristic horizontal length scale is L. The central latitude of the domain is θ_O and R is the earth's radius. The first parameter setting in (2.1.1) implies that inertial accelerations are small compared with the Coriolis acceleration and the second that the variation of the Coriolis parameter is small on the scale L compared to f_O. Quasi-geostrophic theory supposes that

$$b \approx \varepsilon, \ or \ \ \frac{U}{\beta L^2} = O(1) \tag{2.1.2}$$

so that the *gradient* of relative vorticity is of the same order as the *gradient* of the planetary vorticity while the relative vorticity itself is small with respect to f. Similarly, the frictional forces are assumed to be small compared to the Coriolis acceleration.

We choose a local coordinate system in which x is directed eastward, y northward and z upward. The corresponding velocities relative to the rotating earth are (u,v,w).

The density field is partitioned between a density field that is imagined to exist in the absence of motion, $\rho_s(z)$ and a much smaller part which is related to the horizontal pressure gradients and so to the motion. The total density field is therefore,

$$\rho_* = \rho_s(z) + \rho(x,y,z,t), \ \ \rho << \rho_s \tag{2.1.3}$$

Similarly the pressure field is partitioned as,

$$p_* = p_s(z) + p(x,y,z,t), \tag{2.1.4}$$

The hydrostatic approximation is assumed valid and as a consequence for the s-subscripted pressure and density,

$$\rho_s g = -\frac{\partial p_s}{\partial z}$$

while similarly, (2.1.5 a,b)

$$\rho g = -\frac{\partial p}{\partial z}.$$

To lowest order then, the above approximations lead to the geostrophic and hydrostatic approximations which, to an error of O(*ε*) yield,

$$-\rho_s u f_o = \frac{\partial p}{\partial y},$$

$$\rho_s v f_o = \frac{\partial p}{\partial x}, \qquad (2.1.6\ a,b,c)$$

$$-\rho g = \frac{\partial p}{\partial z}$$

To the same order the continuity equation becomes,

$$\rho_s\left[u_x + v_y\right] + \left(\rho_s w\right)_z = 0 \qquad (2.1.7)$$

Here, subscripts are used to denote differentiation. Note that we have not assumed that the background density field $\rho_s(z)$ has negligible variation over the depth *D*, of the fluid as indeed it does not for the case of the atmospheric troposphere where that variation is order one. For the ocean the total variation of $\rho_s(z)$ is small, on the order of one part in a thousand and its variation in (2.1.7) can be ignored. In either case the geostrophic velocities from (2.1.6 a,b) satisfy,

$$u_x + v_y = 0 \qquad (2.1.8)$$

Thus, the *geostrophic* velocity is horizontally non divergent. This implies in turn that the variation of $\rho_s w$ with height is due only to the small, *O(ε)* departures from geostrophy. If the vertical velocity vanishes at some depth, e.g. the sea surface, then it must be small and of *O(ε)* for all z. This eliminates the vertical velocity from the lowest order representation of the advective derivative.

It follows from all this that the lowest order vorticity equation, i.e. the equation for the vertical component of vorticity is:

$$\overbrace{\zeta_t + u\zeta_x + v\zeta_y + \beta v}^{geostrophic\ velocities} + \underbrace{f_o(u_x + v_y)}_{ageostrophic} = curl\Im \tag{2.1.9}$$

In (2.1.9) the relative vorticity

$$\zeta = v_x - u_y \tag{2.1.10}$$

is evaluated using the geostrophic relations (2.1.1 a,b) and the advection of the vorticity is also by the geostrophic velocity as indicated in (2.1.9). On the other hand the induction of planetary vorticity by the horizontal convergence of velocity in the presence of the background planetary vorticity must be due to the ageostrophic velocity, i.e. the departures of the velocity from geostrophic balance since the geostrophic velocities are horizontally non divergent (2.1.8). The term on the right hand side of (2.1.9) is the vertical component of the curl of the horizontal frictional forces.

Note that in (2.1.9) there are several terms missing from the vorticity equation. Consistent with the geostrophic approximation the induction of relative vorticity by the relative vorticity itself $\zeta(u_x + v_y)$ is ignored compared to the planetary term. Similarly, the so-called twisting terms $\xi w_x + \eta w_y$ where ξ and η are the x and y components of relative vorticity are also ignored. In each case the neglected terms are $O(\varepsilon)$ smaller than those retained. The student should check that assertion. Most importantly, the vertical advection of relative vorticity $w\zeta_z$ is absent from the advective term since *w* is small beyond the geometrical factor *D/L* by an additional factor of Rossby number, reflecting the horizontal non divergence of the geostrophic velocity. This gives a two-dimensional flavor to a three dimensional problem since it is only horizontal advection that enters (2.1.9). The continuity equation, (2.1.7) allows us to eliminate the horizontal divergence of the ageostrophic velocity in terms of the vertical velocity,

$$\overbrace{\zeta_t + u\zeta_x + v\zeta_y + \beta v}^{geostrophic\ velocities} = \frac{f_o}{\rho_s}\frac{\partial(\rho_s w)}{\partial z} + curl\Im \tag{2.1.11}$$

What remains to be done is to relate the vertical velocity to the horizontal geostrophic velocity since if that is accomplished we can write (2.1.11) entirely in terms of the pressure field using the geostrophic and hydrostatic relations in (2.1.1).

The vertical velocity in the vortex tube stretching term on the right had side of (2.1.11) will therefore couple the momentum dynamics to the thermodynamics of the problem. That coupling will allow us to eliminate the explicit reference to the vertical velocity.

For example, for the atmosphere where we can use the perfect gas law

$$p = \rho RT \tag{2.1.12}$$

we can take for the thermodynamic equation the statement of the first and second laws of thermodynamics in the form,

$$T\frac{ds}{dt} = Q \tag{2.1.13}$$

where s is the entropy per unit mass and Q is a representation of all the irreversible, non-adiabatic heating of the fluid. For a perfect gas,

$$ds = c_v \frac{dp}{p} - c_p \frac{d\rho}{\rho} \equiv c_p \frac{d\vartheta}{\vartheta} \tag{2.1.14}$$

for any infinitesimal state transformation. The specific heats at constant volume and constant pressure are c_v and c_p respectively. The variable ϑ is the potential temperature and is defined by (2.1.14). For a perfect gas it follows that

$$\vartheta = T\left(\frac{p_{oo}}{p}\right)^{R/c_p} \tag{2.1.15}$$

where p_{oo} is a constant, usually taken to be the pressure at the atmosphere's lower boundary. Note that,

$$s = c_p \ln(\vartheta) + const. \tag{2.1.16}$$

so that,

$$c_p \frac{T}{\vartheta}\frac{d\vartheta}{dt} = Q \tag{2.1.17}$$

Also, it follows from the above relations that,

$$c_p \ln \vartheta = c_v \ln p - c_p \ln \rho \tag{2.1.18}$$

Thus, if we partition the potential temperature as we did the density and pressure,

$$\vartheta_* = \vartheta_s(z) + \vartheta(x,y,z,t) \tag{2.1.19}$$

it follows that,

$$c_p \frac{\vartheta_{s_z}}{\vartheta_s} = c_v \frac{p_{s_z}}{p_s} - c_p \frac{\rho_{s_z}}{\rho_s} \tag{2.1.20}$$

while for the small departures from the standard, rest atmosphere,

$$\frac{\vartheta}{\vartheta_s} = \frac{c_v}{c_p}\frac{p}{p_s} - \frac{\rho}{\rho_s} \tag{2.1.21}$$

Repeated use of the hydrostatic approximation for both the basic pressure and density fields as well as the small departures yields,

$$\frac{\vartheta}{\vartheta_s} = \frac{1}{p_{s_z}}\left[\frac{c_v}{c_p}\frac{p_{s_z}}{p_s} p - p_z\right]$$

$$= \frac{1}{p_{s_z}}\left[\frac{c_v}{c_p}\left(\frac{\vartheta_{s_z}}{\vartheta_s} + \frac{\rho_{s_z}}{\rho_s}\right)p - p_z\right] \tag{2.1.22}$$

$$= \frac{1}{p_{s_z}}\left[\frac{c_v}{c_p}\left(\frac{\vartheta_{s_z}}{\vartheta_s}\right)p - \rho_s (p/\rho_s)_z\right]$$

Using again the hydrostatic relation,

$$g\frac{\vartheta}{\vartheta_s} = \overbrace{\frac{\partial}{\partial z}\left(\frac{p}{\rho_s}\right)}^{A} - \overbrace{\frac{p}{\rho_s}\frac{\vartheta_{s_z}}{\vartheta_s}}^{B} \tag{2.1.23}$$

The ratio of term B to term A is of the order of the proportional change of the basic state potential temperature over the depth of the motion, say the troposphere to the change over the same vertical distance of the change in the pressure field associated with the motion. That ratio is very small in the atmosphere. That is the ratio is of the order,

$$D\frac{\vartheta_{s_z}}{\vartheta_s} << 1$$

where D is the vertical scale of the motion and hence the scale of p. The above ratio is less than 0.1 in the atmosphere. Hence to a good approximation we can write,

$$g\frac{\vartheta}{\vartheta_s} = \frac{\partial}{\partial z}\left(\frac{p}{\rho_s}\right) \tag{2.1.24}$$

which is a convenient replacement for the hydrostatic relation (2.1.6) involving density since it is the equation for the potential temperature we will use to eliminate w. In terms of the potential temperature it follows that the thermal wind equations become,

$$f_o v_z = g\frac{\vartheta_x}{\vartheta_s}$$

(2.1.25 a,b)

$$f_o u_z = -g\frac{\vartheta_y}{\vartheta_s}$$

with the aid of the geostrophic relations and (2.1.24).

To the same order of approximation the thermodynamic equation, (2.1.17) becomes,

$$\vartheta_t + \overbrace{u\vartheta_x + v\vartheta_y}^{\text{geostrophic velocities}} + w\vartheta_{s_z} = \frac{\vartheta_s}{c_p T_s} Q \qquad (2.1.25)$$

In (2.1.25) T_S is the background temperature field and it is also a function only of z.

Note that in (2.1.25) the vertical velocity is retained in the advection term even though it is an order Rossby number smaller than the geostrophic velocities because it is multiplied by the much larger vertical gradient of the background potential temperature ϑ_{s_z} which is an order ε^{-1} larger than the horizontally variable part of the potential temperature. It is analogous to the retention of *w* in (2.1.11) because it is there multiplied by the Coriolis parameter which is also larger, by the same order, than the relative vorticity. Hence each term in its equation is the same order in Rossby number as the terms retained. We can use (2.1.25) to solve for *w,*

$$w = -\left\{\frac{\vartheta_t + u\vartheta_x + v\vartheta_y}{\vartheta_{s_z}}\right\} + \frac{\vartheta_s}{c_p T_s}\frac{Q}{\vartheta_{s_z}} \qquad (2.1.26)$$

We can now calculate

$$(\rho_s w)_z = -\left\{\frac{\rho_s}{\vartheta_{s_z}}\left(\vartheta_t + u\vartheta_x + v\vartheta_y\right)\right\}_z + \left\{\frac{\rho_s Q \vartheta_s}{c_p T_s \vartheta_{s_z}}\right\}_z$$

$$= -\frac{d}{dt}\left[\frac{\partial}{\partial z}\left(\frac{\rho_s}{\vartheta_{s_z}}\vartheta\right)\right] + (u_z \frac{\vartheta_y}{\vartheta_{s_z}} + v_z \vartheta_x)\rho_s + \frac{\partial}{\partial z}(\rho_s H) \tag{2.1.27}$$

where,

$$H = \frac{Q\vartheta_s}{c_p T_s \vartheta_{s_z}} \tag{2.1.28}$$

With the thermal wind relation the middle term in the last line of (2.1.27) vanishes so that

$$\frac{1}{\rho_s}(w\rho_s)_z = -\frac{d}{dt}\left[\frac{1}{\rho_s}\frac{\partial}{\partial z}\frac{\rho_s}{\vartheta_{s_z}}\vartheta\right] + \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s H) \tag{2.1.29}$$

so that the vorticity equation can be written as,

$$\frac{d}{dt}\left[\zeta + \beta y + \frac{f_o}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s \vartheta}{\vartheta_{s_z}}\right)\right] = curl\Im + \frac{f_o}{\rho_s}\frac{\partial}{\partial z}(\rho_s H) \tag{2.1.30}$$

where we recall that the advective derivative to lowest order in Rossby number is,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \tag{2.1.31}$$

because of the smallness of w. This implies that the advective derivative to this order does not affect any variable that is a function only of z.

The geostrophic relations (2.1.6 a,b) suggest that the pressure field acts as a streamfunction for the horizontal motion. We make that identification explicit by defining,

$$\psi = \frac{p}{\rho_s f_o}, \Rightarrow u = -\psi_y,\ v = \psi_x \qquad (2.1.32)$$

so that

$$\zeta = \psi_{xx} + \psi_{yy} = \nabla_h{}^2 \psi \qquad (2.1.33)$$

The hydrostatic relation (2.1.24) yields

$$\vartheta = \vartheta_s \frac{f_o}{g} \psi_z \qquad (2.1.34)$$

Therefore every term in (2.1.30) can be written in terms of the geostrophic stream function,

$$\frac{d}{dt}\left[\nabla_h{}^2 \psi + \frac{1}{\rho_s}\left(\frac{\rho_s f_o{}^2}{N^2}\psi_z\right)_z\right] + \beta\psi_x = curl\Im + \frac{f_o}{\rho_s}(\rho_s H)_z \qquad (2.1.35)$$

This is the quasi-geostrophic potential vorticity equation and will be the principal tool we will use for out investigations of meso-scale instabilities. In (2.1.35) we have used the definition of the buoyancy frequency appropriate for a compressible atmosphere (see Chapter 6 of Geophysical Fluid Dynamics),

$$N^2 = \frac{g}{\vartheta_s} \vartheta_{s_z} \qquad (2.1.36)$$

Note that :

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \psi_x \frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial x} \tag{2.1.37}$$

For the ocean the background density field ρ_s satisfies,

$$\frac{D}{\rho_s} \frac{\partial \rho_s}{\partial z} << 1 \tag{2.1.38}$$

where D is the vertical scale of the motion. Further the thermodynamic equation, for our purposes can be replaced with the equation for a liquid,

$$\frac{d\rho}{dt} + w\rho_{s_z} = -Q' \tag{2.1.39}$$

where Q' is the non adiabatic contribution to the density change. The hydrostatic approximation in the case of a liquid that satisfies (2.1.38) is simply,

$$\rho g = -p_z \tag{2.1.40}$$

and

$$N^2 = -g \frac{\rho_{s_z}}{\rho_s} \tag{2.1.41}$$

The geostrophic stream function can be defined then as,

$$\psi = \frac{p}{\rho_o f_o} \tag{2.1.42}$$

where ρ_o is a constant, independent of z. Retracing our steps it is easy to see that the governing equation for the geostrophic stream function in this case is again (2.1.35) in which the relation between the pressure and stream function is slightly different, N is defined in terms of density instead of potential temperature and ρ_S in (2.2.35) can be taken to be a constant. Otherwise the same equation holds for both the synoptic scale for the atmosphere and the ocean. It is this equivalence that makes much of the discussion of stability theory valid for both the atmosphere and the oceans.

2.2 Boundary conditions

Consider the lower boundary of the fluid. We imagine there is a think Ekman layer which brings the fluid to rest at the actual surface but that our equation apply in a region above that thin region. Then the matching condition of the interior to the boundary provides us with a boundary condition on the vertical velocity. Using a standard result (see Chapter 4 of GFD) we find that *w* right above the boundary layer can be written as,

$$w = w_E + uh_x + vh_y \tag{2.2.1}$$

Here the Ekman pumping velocity

$$w_E = \left(\frac{A_v}{2 f_o} \right)^{1/2} \zeta(x,y,0) = \delta_E \zeta(x,y,0) \tag{2.2.2}$$

is supplemented by the topographic lifting term; h is the height of the topography from the mean elevation of the bottom and δ_E is the Ekman layer thickness.

At the same time, w is obtainable from (2.1.26). After a bit of algebra, equating the two expressions for w yields the boundary condition for the geostrophic stream function at the bottom (z=0),

$$\frac{f_o}{N^2} \frac{d\psi_z}{dt} + \delta_E \nabla_h{}^2 \psi + J(\psi,h) = H,$$

(2.2.3)

$$J(a,b) \equiv a_x b_y - a_y b_x$$

If there are rigid, vertical side walls, the normal velocity must vanish at those boundaries. If they lie along a latitude circle this implies that the lowest order geostrophic velocity must vanish there.

$$v = \psi_x = 0, \qquad (2.2.4)$$

A similar condition must hold if the boundaries in y are very distant and the disturbances vanish at large positive and negative y.

The boundary condition at an upper boundary will be the same as (2.2.3) if it is a rigid boundary (like the surface of the ocean) except that the term involving the Ekman pumping and the topography will both be absent. If the upper boundary is rigid and can support an Ekman layer the same boundary condition holds if we let $\delta_E \to -\delta_E$ since a low pressure center ($\zeta > 0$) at the upper boundary will pump a negative vertical velocity instead of a positive one.

Note that if D is the vertical scale of the motion and L is the horizontal scale of the motion, the advection of relative vorticity will be the same order as the advection of the stretching term if,

$$L \approx ND / f_o \equiv L_d \qquad (2.2.5).$$

That is, the momentum field will couple with the thermal field when the horizontal length scale is of the order of the deformation radius *when the deformation radius is defined in terms of the vertical scale of the motion (not the domain)*.

2.3 The zonal momentum equation.

We are frequently interested in the stability of zonal currents, i.e. currents which are directed along latitude circles. For this reason the zonal momentum equation if of interest, Of course, to lowest order this is given by the geostrophic relation (2.1.6 b). However, it is necessary in deriving the qg potential vorticity equation (2.1.35) to consider the momentum equations to one higher order in Rossby number. At that order the zonal momentum equation can be written,

$$\overbrace{u_t + uu_x + vu_y - \beta y v}^{geostrophic\ velocities} - \underbrace{f_o}_{agesotrophic\ velocity} v_a = -p_x / \rho_s \tag{2.3.1}$$

using the geostrophic relations it follows that

$$\overbrace{u_t + (uu)_x + (vu)_y - \beta y v}^{geostrophic\ velocities} - \underbrace{f_o}_{agesotrophic\ velocity} v_a = -p_x / \rho_s \tag{2.3.2}$$

If we assume that the fields of motion are periodic in the x-direction, or vanish for large positive and negative x then an x- average, denoted by an overbar yields,

$$\bar{u}_t + \overline{vu}_y - f_o \bar{v}_a = 0. \tag{2.3.3}$$

This follows most directly from (2.3.1) and the fact that the x- average of the geostrophic velocity must vanish (since it is a perfect x derivative). However, the average of the ageostrophic velocity need not vanish.

On a side wall where both the geostrophic velocity and the ageostrophic velocity normal to the wall must vanish we find an additional boundary condition for the x-averaged zonal flow,

$$\bar{u}_t = -\bar{\psi}_{yt} = 0. \tag{2.3.4}$$

Note from (2.3.2) that the mean zonal flow can change either due to the transport of eddy momentum by the geostrophic velocity correlation between u and v or by the Coriolis force exerted by the very weak ageostrophic meridional velocity. In quasi-geostrophic theory these are of the same order. In (2.3.2) and *seq.* we have ignored the role of friction. It is left as an exercise for the student to redo the above calculation in the presence of explicit friction acting on the momentum field.

2.4 Zonal mean equations

Although not all flows of interest are zonal and independent of x such flows, for meteorologically historic reasons, form a paradigm for the stability problem. The approach that has been developed can be applied to other problems with some obvious changes (and difficulties). For oceanography, the stability of currents as the Gulf Stream when flowing in the mid-ocean after separation, has been studied as an idealized zonal flow.

Since the perturbations to the flow that arise naturally are wave-like it is common to divide the total flow into a part which is independent of x and another part which has a zero x average and which we may identify with the perturbation. Care must be taken however, since one of the non linear consequence of the instabilities will be to change the mean, (x-averaged) zonal flow so that part of the perturbation will have be contained in the x-averaged flow.

For example for the zonal velocity:

$$u(x,y,z,t) = \overline{u}(y,z,t) + \tilde{u}(x,y,z,t), \qquad \overline{\tilde{u}} = 0 \tag{2.4.1}$$

An overbar represents the x-average and the tilde quantities a departure from the x-average. Obviously, the x-average of any tilde quantity is zero. It also follows that the *geostrophic* meridional velocity has zero x-average, so that,

$$v = \tilde{v} \text{ (geostrophic } v) \tag{2.4.2}$$

while the small, ageostrophic meridional velocity will have, in general a non-zero x-average.

Since the geostrophic velocity is horizontally non divergent, the *x*-average of the zonal momentum equation, (2.3.2) yields,

$$\frac{\partial \overline{u}}{\partial t} = -\frac{\partial}{\partial y}\overline{uv} + f_o \overline{v}_a \tag{2.4.3}$$

where we have again ignored friction.

The first term on the right hand side of (2.4.3) is evaluated with the geostrophic velocities and since the *x*- average of *v* is zero it follows that (2.4.3) can be written,

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial}{\partial y}\overline{\tilde{u}\tilde{v}} - f_o \bar{v}_a \tag{2.4.4}$$

Thus the mean zonal flow will be changed by the Reynolds stress associated with the momentum flux of the flow components which individually have zero x-average and by the Coriolis force due to the weak, zonal mean of the meridional motion. The resulting change in the zonal flow represents a perturbation to the original, basic state and is part of the perturbation field. Since its calculation in (2.4.4) requires knowledge of the ageostrophic flow it can not be directly calculated within the context of q.g. theory. So we must proceed further.

If we also take the x-average of the equation for the potential temperature (or for the density in the oceanic case), we obtain (again ignoring dissipation for ease),

$$\frac{\partial \bar{\vartheta}}{\partial t} = -\frac{\partial}{\partial y}\overline{\tilde{v}\tilde{\vartheta}} - \bar{w}\vartheta_{s_z} \tag{2.4.5}$$

The zonally averaged potential temperature is changed by the average meridional *heat flux* due to the correlation of perturbations in *v and* ϑ each of which have zero x-average. The zonal average of ϑ is also altered by the x-averaged vertical motion in the field of the background static stability ϑ_{s_z}.

At the same time the zonal average of the continuity equation yields,

$$\frac{\partial \bar{v}_a}{\partial y} + \frac{1}{\rho_s}\frac{\partial \rho_s \bar{w}}{\partial z} = 0 \tag{2.4.6}$$

If we use (2.44) to solve for the mean meridional velocity and (2.4.5) to solve for the mean vertical velocity and insert the result in (2.4.6) we obtain,

$$\frac{\partial}{\partial t}\left[\frac{1}{\rho_s}\left(\frac{\rho_s\overline{\vartheta}}{\vartheta_{s_z}}\right)_z-\frac{1}{f_o}\frac{\partial\overline{u}}{\partial y}\right]=-\frac{\partial}{\partial y}\left[\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\overline{\tilde{v}\tilde{\vartheta}}}{\vartheta_{s_z}}\right)-\frac{1}{f_o}\frac{\partial}{\partial y}\overline{\tilde{v}\tilde{u}}\right] \tag{2.4.7}$$

This is the equation for the rate of change of the zonal average of the potential vorticity $\overline{q}$. Using the geostrophic relations we obtain for $\overline{q}$

$$\begin{aligned}\overline{q}&=\left[\frac{1}{\rho_s}\left(\frac{\rho_s\overline{\vartheta}}{\vartheta_{s_z}}\right)_z-\frac{1}{f_o}\frac{\partial\overline{u}}{\partial y}\right]\\&=\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\rho_s\frac{f_o^{\,2}}{N^2}\frac{\partial\overline{\psi}}{\partial z}\right)+\frac{\partial^2\overline{\psi}}{\partial y^2}\end{aligned} \tag{2.4.8}$$

(Note that $\overline{\vartheta}=\vartheta_s\dfrac{f_o}{g}\overline{\psi}_z,\quad \overline{u}=-\overline{\psi}_y$)

Now consider the partition of the potential vorticity into its x-average, given above and its departure from the mean

$$\tilde{q}=\tilde{v}_x-\tilde{u}_y+\frac{f_o}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\tilde{\vartheta}}{\vartheta_{s_z}}\right) \tag{2.5.8}$$

It follows that the meridional flux of potential vorticity due to the geostrophic perturbations with zero zonal mean is

$$\begin{aligned}\overline{\tilde{v}\tilde{q}}&=-\left(\overline{\tilde{v}\tilde{u}}\right)_y+\frac{f_o}{\rho_s}\overline{\tilde{v}\left(\frac{\rho_s\tilde{\vartheta}}{\vartheta_{s_z}}\right)_z}\\&=-\left(\overline{\tilde{v}\tilde{u}}\right)_y+\frac{f_o}{\rho_s}\left(\frac{\overline{\tilde{v}\tilde{\vartheta}}}{\vartheta_{s_z}}\right)_z-\frac{f_o}{\rho_s}\frac{\overline{\tilde{\vartheta}\tilde{v}_z\rho_s}}{\vartheta_{s_z}}\end{aligned} \tag{2.5.9}$$

but because of the thermal wind relation,

$$\overline{\tilde{v}_z \tilde{\vartheta}} = \vartheta_s \frac{g}{f_o} \overline{\tilde{\vartheta}_x \tilde{\vartheta}} \equiv 0 . \tag{2.5.10}$$

Hence (2.4.7) with (2.4.8) and (2.5.9) can be written

$$\frac{\partial \overline{q}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{\rho_s} \frac{\partial}{\partial z}\left(\frac{f_o{}^2 \rho_s}{N^2} \frac{\partial \overline{\psi}}{\partial z} \right) + \frac{\partial^2 \overline{\psi}}{\partial y^2} \right] = -\frac{\partial \overline{\tilde{v}\tilde{q}}}{\partial y}$$

$$\overline{\tilde{v}\tilde{q}} = \frac{\partial}{\partial y} \overline{\tilde{\psi}_x \tilde{\psi}_y} + \frac{1}{\rho_s} \frac{\partial}{\partial z} \frac{f_o{}^2}{N^2} \overline{\rho_s \tilde{\psi}_x \tilde{\psi}_z} \tag{2.5.11}$$

This important result could have been obtained directly from the x-average of (2.1.35) after partitioning between mean and deviation in x. I have chosen to do it this way to emphasize the following important point. Although the momentum equation and the heat (density) equation each contain x-averages of quantities that cannot be directly calculated from quasi-geostrophic theory, the x-averaged potential vorticity equation can be expressed entirely in terms of the geostrophic stream function. If we know the tilde quantities (eddies) we can calculate the fluxes and from (2.5.11) calculate directly the time rate of change of the zonally averaged potential vorticity $\overline{q}$. Once $\overline{q}$ is known we can invert the elliptic operator (2.4.8) relating the stream function and the potential vorticity. Once we have the zonally averaged stream function we can calculate directly the rate of change of the mean zonal velocity and the associated mean potential temperature (density) without explicit reference to the ageostrophic velocity. Within quasi-geostrophic theory this is the most straight forward way to obtain the effect of the eddies on the mean flow.