Chapter 12. Barotropic Instability

12.1 Formulation

We have concentrated on the baroclinic instability problem in which the source of energy is the potential energy made available by the horizontal density gradients in the fluid. There is a second source of energy for instabilities manifested in the horizontal shear of the current which may be released by the Reynolds stresses. This kind of instability is called barotropic instability because it can occur in a non stratified or barotropic fluid. However, the process can also occur in a stratified fluid and can coexist with baroclinic instability. However, to study its properties in the purest and simplest form first consider a basic current which has horizontal but not vertical shear. That is, let

$$U_o = U_o(y) \tag{12.1.1}$$

We will also consider only a flat bottom. The normal mode equation (5.1.4) for such a basic flow simplifies to,

$$(U_o - c)\left[\frac{1}{\rho_s}\frac{\partial}{\partial z}\frac{\rho_s f_o^{\,2}}{N^2}\frac{\partial \Phi}{\partial z} + \frac{\partial^2 \Phi}{\partial y^2} - k^2 \Phi\right] + \Phi \frac{\partial q_o}{\partial y} = 0,$$

$$(U_o - c)\Phi_z = 0, \quad z = 0, D \qquad (12.1.2a,b,c,d)$$

$$\Phi = 0, \; y = y_1, y_2$$

$$\frac{\partial q_o}{\partial y} = \beta - U_{o_{yy}}$$

In the absence of topography the simple boundary condition on the lateral boundaries allows a separation of vertical and horizontal structure of the problem. Consider the Sturm-Liouville problem,

$$\frac{1}{\rho_s}\frac{d}{dz}\left(\frac{\rho_s f_o^{\,2}}{N^2}\frac{dZ_j}{dz}\right) = -\lambda_j Z_j,$$

(12.1.3.a,b)

$$\frac{dZ_j}{dz} = 0, \quad z = 0, D$$

This standard eigenvalue problem generates a complete set of eigenfunctions and eigenvalues $\{Z_j, \lambda_j\}$ which can be used to represent an arbitrary vertical structure for the perturbation. That is, we could write any perturbation as,

$$\Phi = \mathrm{Re} \sum_{j=0} A_j(y) Z_j(z) e^{ik(x - c_j t)} \qquad (12.1.4)$$

and for each term in the sum the governing equation is

$$[U_o - c]\left[\frac{d^2 A}{dy^2} - (\lambda + k^2) A\right] + \left(\beta - U_{o_{yy}}\right) A = 0,$$

(12.1.5 a,b)

$$A = 0, \quad y = y_1, y_2$$

where we have suppressed the subscript *j* on *A*, λ and *c*. Note that the lowest eigenvalue for the problem (12.1.3) is always λ=0 corresponding to an eigenfunction independent of z (and without loss of generality we may take it as 1). This is furthermore independent of the structure of *N(z)*.

We define the total wavenumber as $\kappa^2 = k^2 + \lambda$. It is clear from (12.1.5a) that the phase speed *c* will be a function only of κ. On the other hand, the growth rate is

$$\omega_i = k c_i(\kappa) \qquad (12.1.6)$$

For a given κ the growth rate will be largest for the maximum possible *k* at that κ and that clearly takes place for the eigenvalue corresponding to λ=0. For a stratified fluid subject to the boundary condition (12.1.2b) a basic flow which is independent of z will be most unstable to perturbations that are also independent of z. We will henceforth consider only barotropic disturbances.

The necessary condition for instability for the barotropic flow is simply,

$$c_i \int_{y_1}^{y_2} |A|^2 \frac{(\beta - U_{oyy})}{|U_o - c|^2} dy = 0 \qquad (12.1.7)$$

Therefore, the potential vorticity gradient $\partial q_o / \partial y$ must change sign. A weak shear will always be stabilized by β. If the perturbations are independent of z there is no density anomaly produced and no vertical motion. In particular, there is no x-averaged ageostrophic meridional velocity and the x-averaged momentum equation reduces to,

$$\frac{\partial}{\partial t}\overline{u} = -\frac{\partial}{\partial y}\overline{u'v'} = \overline{v'q'} \qquad (12.1.8)$$

at each z level. Similar to our argument in the baroclinic case we can relate the potential vorticity perturbation to the meridional displacement η,

$$q' = -\eta \frac{\partial q_o}{\partial y}$$

$$= -\frac{A}{U_o - c}\frac{\partial q_o}{\partial y} \qquad (12.1.9\ a, b)$$

so that since

$$v' = \frac{ik}{2}\left[Ae^{i\theta} - A^* e^{-i\theta^*}\right]$$

$$q' = \frac{1}{2}\left[-\frac{A}{U_o - c}e^{i\theta} - \frac{A^*}{U_o - c^*}e^{-i\theta^*}\right]\frac{\partial q_o}{\partial y} \qquad (12.1.10\ a,b)$$

it follows that,

$$\overline{v'q'} = -\frac{kc_i}{2} \frac{|A|^2}{|U_o - c|^2} \frac{\partial q_o}{\partial y} e^{2kc_i t} \tag{12.1.11}$$

Therefore (12.1.8) becomes,

$$\frac{\partial \overline{u}}{\partial t} = -\frac{kc_i}{2} \frac{|A|^2}{|U_o - c|^2} \frac{\partial q_o}{\partial y} e^{2kc_i t} \tag{12.1.12}$$

the integral of which, over the y domain must vanish if momentum is conserved, leading to (12.1.7). The implication for the instability of thin barotropic jets is immediate as Figure 12.1.1 shows.

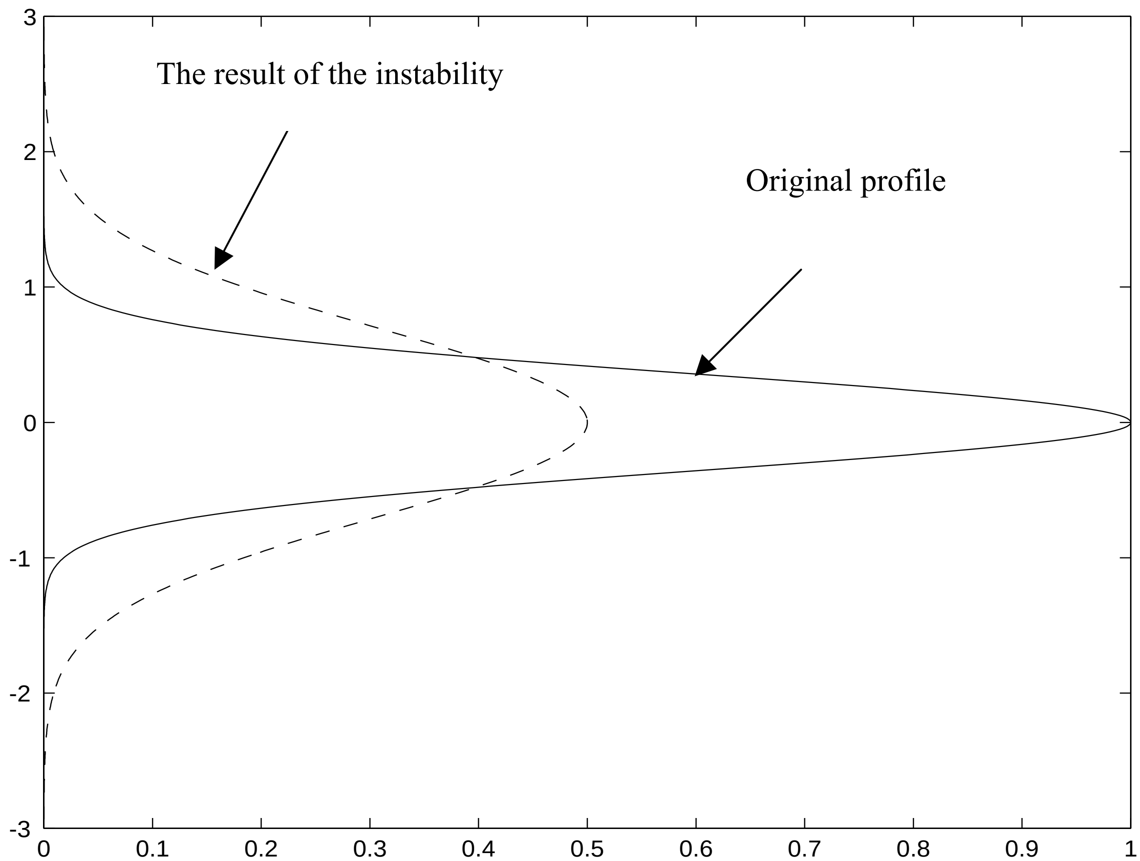

Figure 12.1.1 The thin jet is barotropically unstable and as a result becomes broader.

At the jet center the profile curvature $U_{oyy} < 0$ and so , from (12.1.2) a growing disturbance must produce an Reynolds stress momentum flux divergence which reduces the jet velocity at the core. On the flanks of the jet where U_{oyy} changes sign the potential vorticity gradient becomes positive and the local zonal momentum increases (the total, after all must remain unchanged). The effect of the barotropic instability is to broaden , and weaken the jet, *qualitatively* analogous to the action of momentum diffusion. We recall that for the baroclinic problem, where the energy source was the potential energy of the basic state, an instability would generally *sharpen* the jet profile. The two mechanisms can be thought of as providing a balancing scheme. A jet which is very broad but with vertical shear will become unstable and reduce the vertical shear but sharpen the jet. A sharp jet becomes barotropically unstable producing a broader jet.

Consider the limit as $c_i \rightarrow 0$ from above. That is, consider a neutral mode that is contiguous to a barely unstable mode. The Reynolds stress gradient ,

$$-\frac{\partial}{\partial y}\overline{u'v'} = -\frac{kc_i}{2}\frac{|A|^2}{|U_o - c|^2} = -\frac{k}{2}\mathrm{Im}\frac{|A|^2}{(U_o - c)} \qquad (12.1.13)$$

As $c_i \rightarrow 0$ the Reynold stress convergence goes to zero *except where* $U_o - c \rightarrow 0$, i.e. at the critical layer at $y=y_c$. If we integrate (12.1.13) across that value of y, using the second form of (12.1.13) the calculus of residues yields a jump in the momentum flux at the critical level.

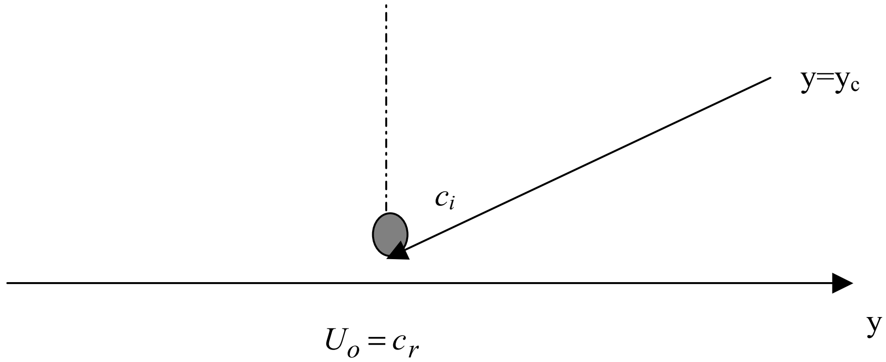

Figure 12.1.2 The singularity in the complex y plane at the critical level y=y_c. For $c_i > 0$ the singularity lies slightly above the real y line. The dashed line shows the branch cut associated with the logarithmic singularity at the critical point.

The integral of the momentum flux, as $c_i \rightarrow 0$ must make a detour below the singularity and this picks up *half* the residue of the simple pole as a contribution. Thus there is a jump in the Reynolds stress,

$$\overline{-u'v'}\ \Big) = -k\pi \frac{|A|^2}{2U_o'} \frac{\partial q_o}{\partial y}\Big)_{y=y_c} \qquad (12.1.14)$$

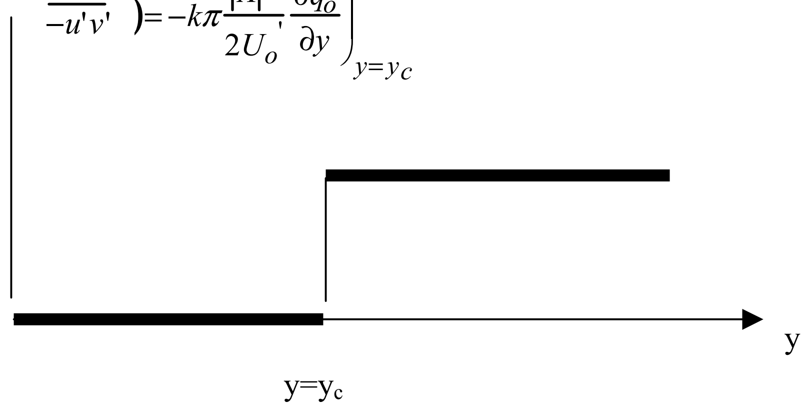

Figure 12.1.3 The jump in the Reynolds stress as the critical level is passed in the limit $c_i \rightarrow 0$.

If $\overline{u'v'}$ vanishes at the end points, y = y_1, y_2 there is clearly a contradiction (although if one of these points is at infinity and radiation is allowed there may be a way out of this contradiction). For finite regions in which the Reynolds stress must vanish at each boundary it follows that the only way we can have a neutral mode that is contiguous to an unstable mode is if *the potential vorticity gradient vanishes at the critical layer*. Since the potential vorticity gradient must vanish for instability this implies that the marginally stable wave, contiguous to an unstable wave, must have its critical layer at the zero of the pv gradient. In other words we can find the phase speed of the important marginally stable wave by identifying it with the basic flow velocity at the point y_c where $\partial q_o / \partial y = 0$, i.e.

$$c = U_o(y_c), \quad \frac{\partial q_o}{\partial y}(y_c) = 0 \tag{12.1.14}$$

This, is we shall see yields a regular eigenvalue problem for the wave number of the marginally unstable mode.

Define,

$$K(y) = \frac{\beta - U_{o_{yy}}}{U_o - c} \tag{12.1.15}$$

and we suppose that for the marginally stable wave *K* is positive (this is a version of the Fjörtoft theorem). The condition (12.1.14) implies that *K(y)* is everywhere finite (recall that *c* is known for the marginally stable wave).This yields a standard Sturm-Liouville eigenvalue problem for the wave number *k*, i.e. from (12.1.5 a)

$$A_{yy} + \left(K(y) - k^2\right)A = 0 \tag{12.1.16}$$

Note that for the neutral mode *A* can be taken to be real without loss of generality. If one multiplies (12.1.6) by *A*, and integrates over the whole y interval, an integration by parts and the use of the boundary conditions (12.1.5b) yields,

$$k^2 = \frac{\int_{y_1}^{y_2} K(y)A^2 dy - \int_{y_1}^{y_2} {A_y}^2 dy}{\int_{y_1}^{y_2} A^2 dy} \tag{12.1.17}$$

It is easy to show, and is a standard result of Sturm-Liouville theory that the differential equation (12.1.16) is equivalent to the variational problem that arise from the condition that k^2 be stationary with respect to variations in *A*. This provides us with a useful approximation method to determine the wavenumber of marginal stability. However, it also yields an important bound on the instability. Since *A* vanishes at the end points of the interval, our earlier results of Chapter 4, namely,

$$\int_{y_1}^{y_2} A_y{}^2 dy \geq \frac{\pi^2}{(y_1 - y_2)^2} \int_{y_1}^{y_2} A^2 dy \tag{12.1.18}$$

so that, (12.1.17) implies,

$$k^2 \leq K_{\max} - \left[\pi^2 / (y_1 - y_2)^2\right] \tag{12.1.19}$$

Thus,

$$K_{\max} \geq \frac{\pi^2}{(y_1 - y_2)^2} \tag{12.1.20}$$

is a condition for the existence of a marginally stable wave, and by implication, of instability. Even if the potential vorticity gradient vanishes in the y-interval, a too narrow interval will render the flow stable. (Reminiscent of the result of the Eady problem).

Suppose *K* is large enough to satisfy (12.1.20) and imagine that we have found the marginally stable solution corresponding to c_i just equal to zero at a $k = k_o$ as the solution to (12.1.16). Consider a perturbation of the solution for slightly different wave number, such that,

$$k = k_o + \varepsilon k_1, \quad \Rightarrow k^2 = k_o{}^2 + 2\varepsilon k_o k_1 + O(\varepsilon)^2 \tag{12.1.21a}$$

We similarly expand *A* and *c,*

$$A = A_o + \varepsilon A_1 + \ldots$$

(12.1.21 b,c)

$$c = c_o + \varepsilon c_1 + \ldots$$

Insertion into (12.1.5a) leads at lowest order to (12.1.16). At O(ε) we obtain,

$$(U_o - c)\left[A_{1_{yy}} - k_o^2 A_1\right] + K(y)A_1(U_o - c) =$$

$$c_1\left\{A_{o_{yy}} - k_o^2 A_o\right\} + 2k_1k_o(U_o - c_o)A_o$$

$$= -c_1 KA_o + 2k_1k_o(U_o - c)A_o$$

(12.1.22)

If we multiply (12.1.22) by $A_o/(U_o - c)$, integrate over the y interval an integration by parts shows that the left hand side integrates to zero, leaving as a condition for solution,

$$c_1 \int_{y1}^{y2} \frac{KA_o^2}{(U_o - c_o)} dy = 2k_1k_o \int_{y1}^{y2} A_o^2 dy \qquad (12.1.23)$$

If K >0 there is a singularity in the integral on the left hand side at the critical layer. We interpret the integral in the same way we did the integral in (12.1.13) that is,

$U_o - c_o = \lim_{c_i \to 0} U_o - c_o - ic_i$ so that the integral on the left hand side consists of two terms,

$$[\Re + i\aleph], \qquad \Re = P\int \frac{KA_o^2}{(U_o - c_o)} dy, \qquad \aleph = \frac{\pi K(y_c)A_o^2(y_c)}{U_o'(y_c)} \qquad (12.1.24\text{ a,b})$$

here $\Re$ is real and is the Cauchy principal part of the singular integral while $\aleph$ is the magnitude of the imaginary part which derives from the residue of the singularity of the integral at the critical point. Then (12.1.23) implies that,

$$\text{Im}c_1 = \frac{-2k_1k_o \int_{y1}^{y2} A_o^2 dy}{\Re^2 + \aleph^2}\left[\frac{\pi K(y_c)A_o^2}{U_o'(y_c)}\right] \qquad (12.1.25)$$

If, as supposed K is positive, then if the critical layer occurs at a point of positive shear, the imaginary part of ic_1 will be positive (instability) if $k_1 < 0$, i.e for slightly longer wavelengths. Thus k_o is a critical wavenumber bounding unstable modes at longer wavelengths. The situation, as we shall see becomes more complicated if the region is not finite so that the Reynolds stress need not vanish on one of the (infinite) boundaries.

12.2 The cosine jet example

H.L. Kuo examined (1973, Advances in Applied Mechanics. Vol. 13, 247-331) the barotropic instability of the "cosine" jet. To facilitate the discussion we introduce dimensionless variables. The jet is contained in a channel defined by $-L \le y \le L$, so L is used to scale all horizontal lengths. The maximum value of the jet velocity is U and it is used to scale the velocity while the ratio L/U is used to scale the time. The resulting equation for the perturbation streamfunction is again (12.1.5) except that now all parameters are nondimensional and $\beta = \beta_{\text{dim}} L^2 / U$ while $-1 \le y \le 1$. In these units the basic velocity Kuo examined is

$$U_o = \frac{1 + \cos \pi y}{2} = \cos^2(\pi y/2) \qquad (12.2.1)$$

and is shown below.

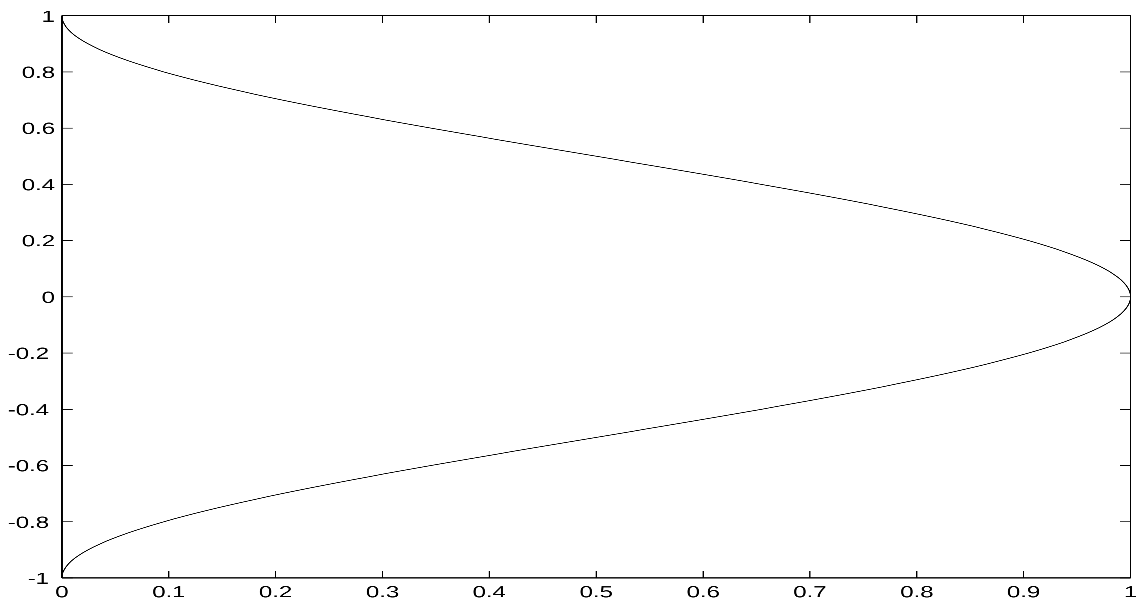

Figure 12.2.1 The cosine jet velocity profile.

The potential vorticity gradient

$$q_{o_y} = \beta - U_{o_{yy}} = \beta + \frac{\pi^2}{2}\cos \pi y \qquad (12.2.2)$$

If $|\beta| \geq \pi^2/2$ the flow is stable. Note the absolute value sign for β. Negative nondimensional β, corresponds to an eastward jet with the same profile. The potential vorticity gradient vanishes at $y = y_c$ where,

$$\cos \pi y_c = -2\frac{\beta}{\pi^2} \qquad (12.2.3)$$

i.e. where

$$U_o = \frac{1}{2} - \frac{\beta}{\pi^2} \qquad (12.2.4)$$

which is the average of the basic flow diminished by an amount reminiscent of the Rossby wave formula. However, (12.2.4) is not a dispersion relation but gives us the basic flow velocity at the position of the vanishing of the pv gradient and hence the phase speed of the marginally stable disturbance. Note that if β were zero, this point occurs at $y_c = \pm 1/2$. As β increases the value of y_c increases until at the critical value of $\beta = \pi^2/2$ it moves to the boundary and then out of the domain.

The ratio,

$$K(y) = \frac{\beta - U_{o_{yy}}}{U_o - c} = \frac{\beta + \frac{\pi^2}{2}\cos \pi y}{\frac{\cos \pi y}{2} + (\frac{1}{2} - c)} = \pi^2 \qquad (12.2.5)$$

for

$$c = U_o(y_c) = \frac{1}{2} - \frac{\beta}{\pi^2} \qquad (12.2.6)$$

This reduces the amplitude equation to a very simple problem for the wavenumber of the marginally stable disturbance, i.e.

$$A_{yy} + \left(\pi^2 - k^2\right)A = 0 \qquad (12.2.7)$$

whose solution,

$$A = \cos(\pi y/2) \qquad (12.2.8)$$

yields, for the wavenumber,

$$k = \sqrt{3}\pi/2 \qquad (12.2.9)$$

Kuo rather ingeniously[♦] was able to find another neutral solution. In this solution $c=0$ and

$$A = [\cos(\pi y/2)]^r \tag{12.2.10}$$

where $r = \left[1-(k/\pi)^2\right]^{1/2}$ (12.2.11)

and

$$\frac{\beta}{\pi^2} = \frac{9}{16} - \left\{\left(1-\frac{k^2}{\pi^2}\right)^{1/2} - \frac{1}{4}\right\}^2 \tag{12.2.12}$$

which is in the nature of a neutral curve. The curves (12.2.12) and (12.2.9) [independent of β] bound the region of instability in the k,β plane. The contours of growth rate within those boundaries were calculated numerically by Kuo and are shown in Figure 12.2.2.

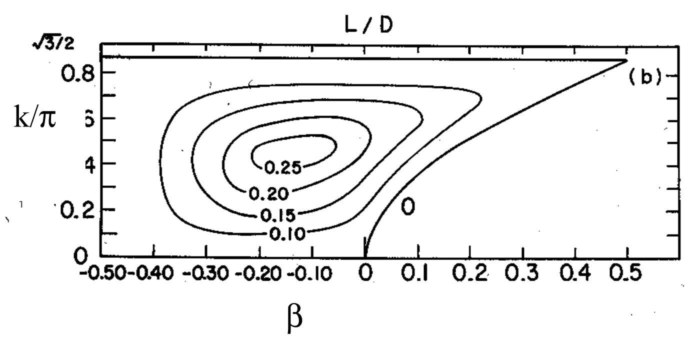

Figure 12.2.2 Contours of growth rate as calculated by Kuo (1973) for the cosine jet.

[♦] This is a generalization of the classical result for the zero β problem where there is always a solution $c=0$, $k=0$, $A=U_o(y)$ to which (12.2.10) reduces in the case $\beta=0$.

Note that easterly flows (negative β) are more unstable than westerly flows. The upper boundary of the unstable domain is given by(12.2.9) and the right hand boundary by (12.2.12).

12.3 "Broken line" velocity profiles

The complexity of the barotropic stability equation and the existence of critical levels has led to the introduction of a class of simpler problems aimed at avoiding these difficulties. In the classical fluid mechanics case when β =0 this has led to the so-called "broken line" velocity profiles. The velocity field is represented by a profile in which $U_o(y)$ is either piece-wise linear in y or constant. For zero β this leads to an equation for which the pv gradient is everywhere zero. The equation reduces to

$$A_{yy} - k^2 A = 0 \qquad (12.3.1)$$

and all the dynamics of the instability is contained in the correct matching conditions at the points where either the velocity or the shear is discontinuous. It is therefore vitally important to get those conditions correctly. To do so we return to the original equation,

$$(U_o - c)\left[A_{yy} - k^2 A\right] + \left(\beta - U_{o_{yy}}\right)A = 0 \qquad (12.3.2)$$

Imagine a velocity profile which is continuous but which in the neighborhood of a point y_o will become increasingly steep such that in the limit either the shear or the velocity itself will become discontinuous. What conditions does (12.3.2) place on the solution A in the neighborhood of that point? We rewrite (12.3.2) as,

$$\frac{\partial}{\partial y}\left\{(U_o - c)A_y - U_{o_y} A\right\} - k^2 (U_o - c)A + \beta A = 0 \qquad (12.3.3)$$

Now consider the integral of (12.3.3) across a small region containing the point y_o as shown in the figure below.

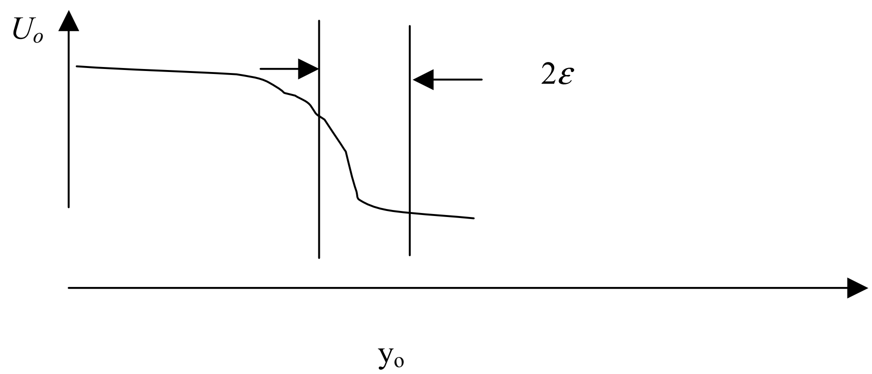

Figure 12.3.1 The velocity profile varying rapidly around the point y_o.

If we integrate (12.3.3) in a small, $O(\varepsilon)$ neighborhood around the point where the velocity may become discontinuous in the limit, we obtain, with errors of $O(\varepsilon)$ (which go to zero as $\varepsilon \rightarrow 0$)

$$(U_o - c)A_y - U_{o_y} A = Y_1(y) \tag{12.3.4}$$

where $Y_1(y)$ is a continuous function at y_o even in the limit where the velocity becomes steep. We understand the each term on the left hand side can be evaluated just before and just after the point, i.e. on either side of any discontinuity in shear or velocity so that the sum of the terms on the left hand side is the same on both sides. A second condition is obtained by noting that (12.3.4) itself can be written,

$$\frac{d}{dy}\left(\frac{A}{U_o - c}\right) = \frac{Y_1(y)}{(U_o - c)^2} \tag{12.3.5}$$

As long as the velocity field remains finite the integral of (12.3.5) implies that

$$\frac{A}{U_o - c} = Y_2(y) \tag{12.3.6}$$

where $Y_2(y)$ is a continuous function. Note that as long as c is complex the integral in (12.3.5) is guaranteed to be finite.

a) The Helmholtz shear layer.

As an example of the use of the broken line profiles, consider the instability of the flow,

$$U_o = \begin{cases} 1, & y > 0 \\ -1, & y < 0 \end{cases} \qquad (12.3.7)$$

This may be thought of as a (heavily) idealized model of a shear layer, in this case with infinitesimal width separating two streams of different constant velocity. Keep in mind that we are using the non-dimensionalization introduced in **12.2**. To keep matters simple to begin with, we will take β to be zero (i.e. small horizontal scales). Then the solution for A is,

$$A = \begin{cases} a_1 e^{-ky} & y > 0 \\ a_2 e^{ky} & y < 0 \end{cases} \qquad (12.3.8)$$

The matching conditions (12.3.4) and (12.3.6), applied at y =0 yield,

$$\begin{aligned} -a_1 k(1-c) &= -a_2 k(1+c) \\ \frac{a_1}{(1-c)} &= -\frac{a_2}{(1+c)} \end{aligned} \qquad (12.3.9\ \text{a,b})$$

Eliminating a_1 and a_2 between (12.3.9 a,b) yields a quadratic equation for c,

$$\begin{aligned} &(1+c)^2 + (1-c)^2 = 0, \\ &\Rightarrow \\ &c = \pm i \end{aligned} \qquad (12.3.10,\ \text{a,b})$$

The phase speed is always complex, i.e. there is instability for all wavelengths. The real part of the phase speed is just the average zonal flow (here zero) and the growth rate for the unstable mode is = k so that the shortest waves grow the fastest, in fact they have unbounded growth rates as k gets larger. One might justifiably object that the shear layer model (12.3.8) is unrealistic for very short waves whose small scales would become increasingly aware of the detailed structure of the shear layer which we have in (12.3.8) collapsed into a discontinuity. Clearly, that idealization should be valid only for scales large compared to the transition region between the two streams. To check that presumption let us reexamine the problem with a somewhat more realistic profile, shown in figure 12.3.2.

b) Finite width shear layer.

We have moved the velocity fields by a Galilean transformation so that its average velocity is now zero. The transition region occurs over a scale *2L*. If we take L as our length scale for the nondimensional coordinates x and y we obtain the profile as shown,

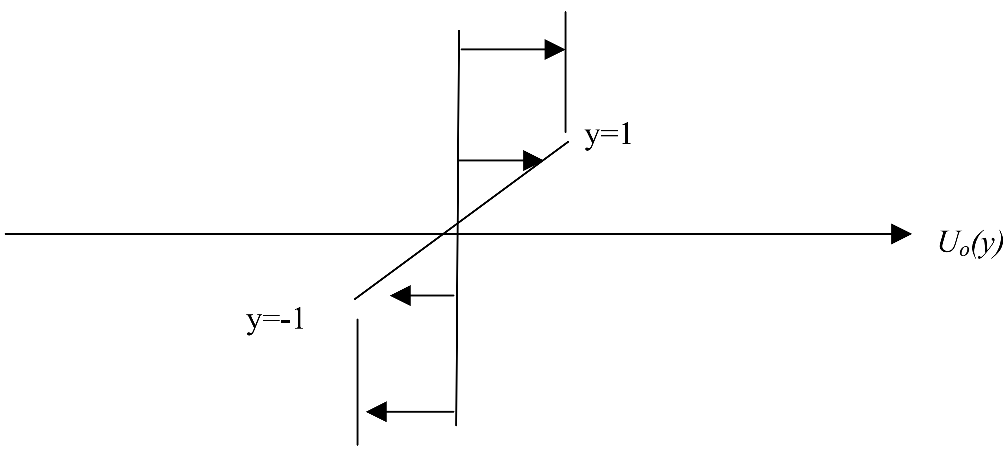

Figure 12.3.2 The shear layer with a finite width.

The basic velocity profile is now,

$$U_o = \begin{cases} 1, & y \geq 1, \\ y & -1 \leq y \leq 1, \\ -1 & y \leq -1 \end{cases} \tag{12.3.11}$$

In our scaling, with the use of L , half the shear layer width as our length scale, the nondimensional wavenumber k is related to the dimensional wavenumber as,

$$k = k_{\dim} L \tag{12.3.12}$$

The limit as $L \rightarrow 0$ will then correspond, for fixed k_{dim} to $k \rightarrow 0$. Or, equivalently, for fixed L, the limit $k \rightarrow 0$ corresponds to very long wavelengths compared to the shear layer thickness. It is in that limit that our previous calculation is relevant, certainly not for large k. The solutions for A can be written,

$$A = \begin{cases} A_1 e^{-k(y-1)}, & y \geq 1, \\ A_2 e^{-k(y-1)} + B_2 e^{k(y+1)}, & -1 \leq y \leq 1, \\ & \\ A_3 e^{k(y+1)} & y \leq -1 \end{cases} \tag{12.3.13}$$

Applying (12.3.6) on y =1 and y =-1 yields,

$$A_1 = A_2 + B_2 e^{2k}$$
$$A_3 = A_2 e^{2k} + B_2$$
(12.3.14 a,b)

while applying (12.3.4) at the same points yields,

$$A_1[1 + k(c-1)] = (c-1)k\left[A_2 - B_2 e^{2k}\right]$$
$$A_3[1 - k(c+1)] = (c+1)k\left[A_2 - B_2 e^{2k}\right]$$
(12.3.15 a,b)

The condition for non trivial solutions of these four linear, homogeneous algebraic equations yields the eigenvalue c as the solution of a quadratic equation, i.e.

$$c^2 = \frac{\left([1-2k]^2 - e^{-4k}\right)}{4k^2} \tag{12.3.16}$$

As $k \rightarrow 0$, which should recover our earlier, zero width shear layer results, both the numerator and denominator vanish and, in fact, in that limit, $c \rightarrow \pm i$ just like our earlier result. However, for large k the exponential term can be ignored and it is clear that c is real. Thus for small enough wavelengths the two kinks in the shear layer are so far apart (compared to a wavelength which is the decay scale of the solution) that they are independent and the flow becomes stable to such small scales (this should be both mathematically and physically reminiscent to you of the small scale stability in the Eady problem). Figure 12.3.3 shows the solution for c and kc_i as a function of wavelength. The critical (dimensionless) wavenumber for instability is approximately $k = k_{\text{dim}}L =$ 0.6272.

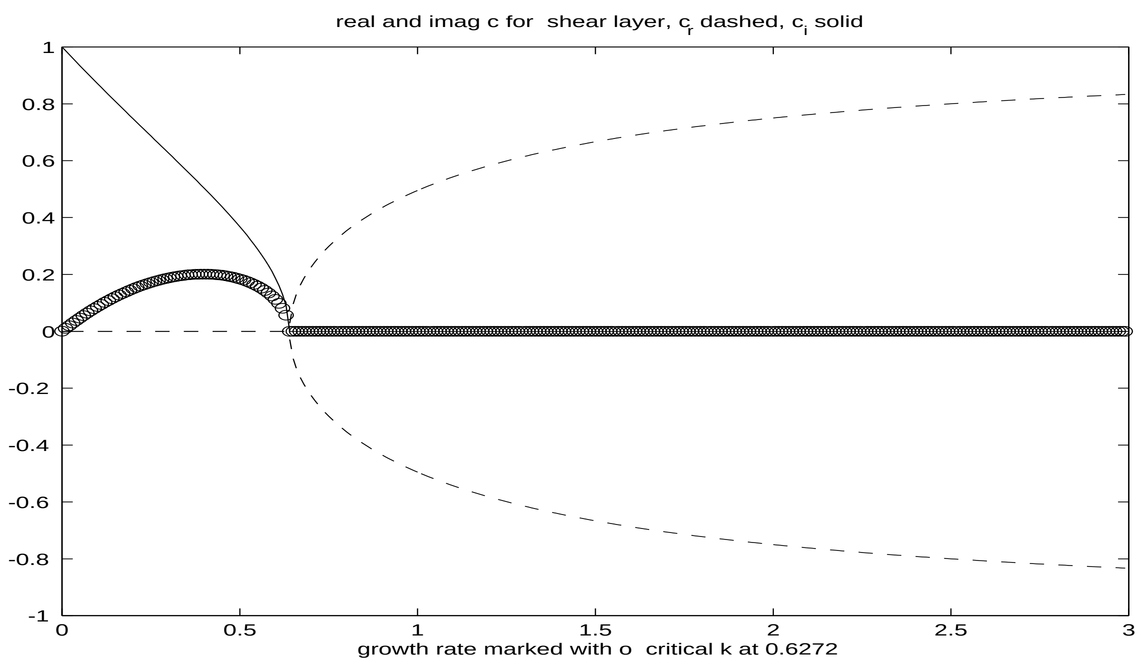

Figure 12.3.3 The real and imaginary (positive root only) for c for the shear layer. The growth rate is shown with the heavy circles.

c) Shear layer with β

Let's examine the effect of β on the stability properties of the shear layer. We will start with the simpler model in which the shear layer thickness is zero. We will do this for two reasons. The principal one is that the problem for the finite layer thickness becomes very complex. Indeed, you can see that it is mathematically similar to the Charney problem, only harder (two boundaries). The second reason is to see whether β is able to stabilize the shear layer zero-thickness shear layer even for long wavelengths.

We examine again the profile given by (12.3.8) but with nonzero β. The equations for A become,

$$A_{yy} - k^2 A + \frac{\beta}{U_o - c} A = 0 \qquad (12.3.17)$$

The solution can be written,

$$A = \begin{cases} a_1 e^{-ly} & y > 0 \\ a_2 e^{l_2 y}, & y < 0 \end{cases} \qquad (12.3.18)$$

where now,

$$l^2 = k^2 + \frac{\beta}{c-1},$$
$$l_2^2 = k^2 + \frac{\beta}{c+1} \qquad (12.3.19\ a,b)$$

The application of the jump conditions at y =0 yield,

$$\frac{a_1}{1-c} = -\frac{a_2}{1+c},$$

$$-l(1-c)a_1 = -l_2(1+c)a_2, \qquad (12.3.20 \text{ a,b,c})$$

$$\Rightarrow$$

$$l(1-c)^2 + l_2(1+c)^2 = 0$$

Since both l and l_2 depend on c the dispersion relation (12.3.20 c) is rather complicated. A bit of algebra yield a cubic equation for c,

$$c^3 + \frac{3\beta}{4k^2}c^2 + c + \frac{\beta}{4k^2} = 0. \qquad (12.3.21)$$

For very small β the solutions are simply $c = \pm i$ and c =0. The first two roots are the solutions previously found for the shear layer. The third root is a spurious root. Notice that if c =0 the wave amplitudes must be zero and hence the solution is trivial. This spurious root arises because we have squared up the form (12.3.20 c) to deal with the square root implied by (12.3.19) in order to obtain (12.3.21). We must be aware of this in examining the roots for nonzero β. There will be only two legitimate roots. For very large β the two roots asymptote to $c = \pm i / 3^{1/2}$ so that although the growth rate is reduced β is unable to stabilize the flow. It is possible to find an analytical solution for the cubic (12.3.21) but the complexity of the solution makes it not very revealing. It is easy to find the roots of (12.3.21) numerically as a function of B= β/k^2 . Figure 12.3.4 shows the positive roots for c and for the parameters l and l_2 (each scaled by k).

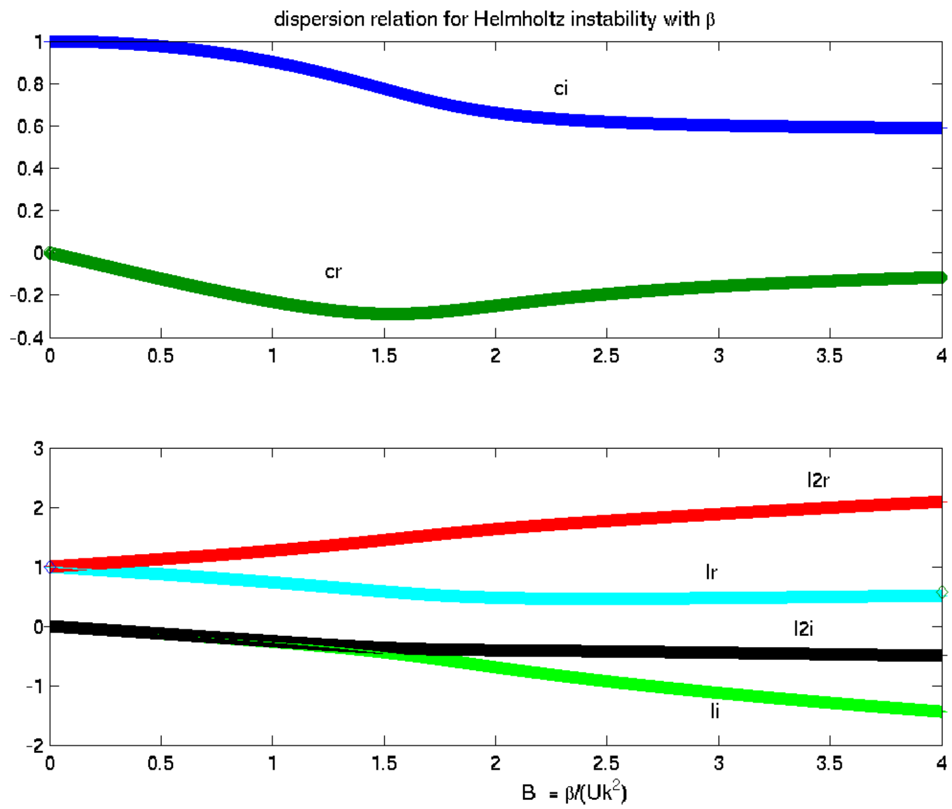

Figure 12.3.4 The upper panel shows the real and imaginary parts of c and the lower panel shows the real and imaginary parts of the decay parameters l and l_2.

We can see the roots for c approaching their proper asymptotic values for large and small values of B. Note that for small B the decay scales in y are simply k and the decay is equal on both sides of the shear region. For larger B the effect of β changes the structure asymmetrically. The decay is more rapid for $y < 0$ than for $y > 0$. For larger B the decay in the negative y direction is rapid. On the other hand the imaginary part of the decay parameters, which represents an oscillatory behavior in y is larger for the positive y region. Indeed, since $l_i >> l_r$ we interpret the region $y > 0$ as containing a wavelike behavior. It is damped in *y* but only after several wavelengths. We will investigate this phenomenon more closely in the next section. The eigenfunction structure at t=0 is shown below,

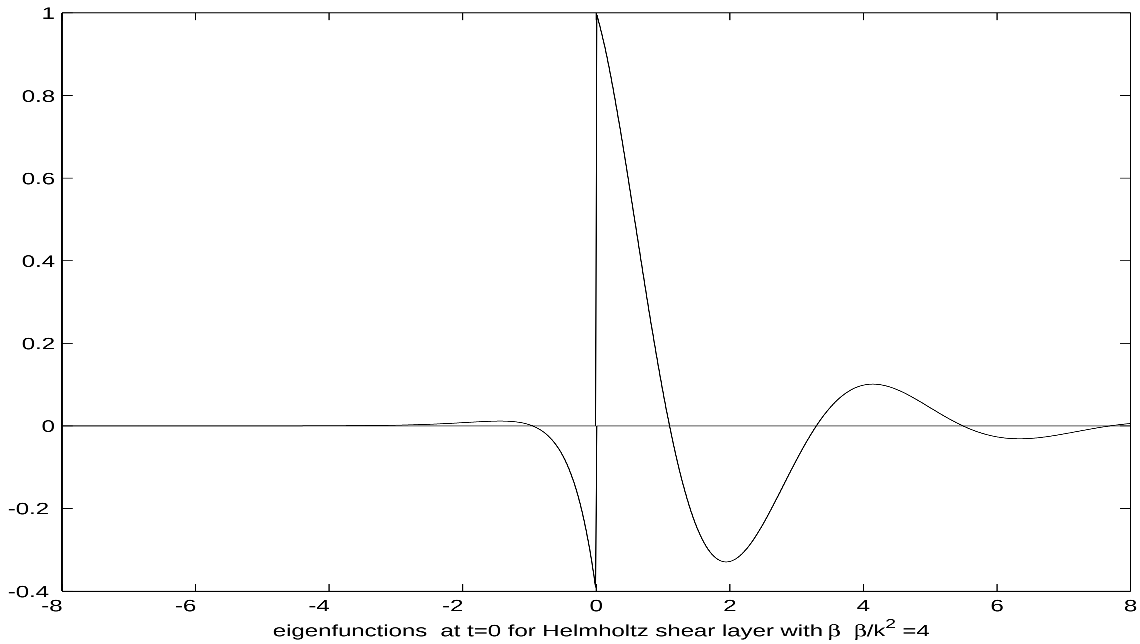

Figure 12.3.5 The form of the eigenfunction for B=4. Note the rapid decay for $y < 0$ and the oscillation for $y > 0$.

12.4 The continuous shear layer and wave radiation.

We noted in section 12.1 that if the perturbation must vanish at the end points of the y interval a neutral disturbance contiguous to an unstable disturbance must have its phase speed equal to the basic flow velocity at the point where the potential vorticity gradient vanishes. This allows us, in those cases, to find the neutral curve in the β,*k* plane as the locus of points corresponding to such modes. However, if the perturbation does not vanish on the boundary there could be a marginally neutral solution whose phase speed is not coincident with the basic flow velocity at the zero of the pv gradient. Such modes would have a singularity at the critical layer. This behavior is possible when the domain is infinite in y and the flow can support neutral, radiating waves at infinity. The purpose of this section is to explore how this changes the stability problem. The example is the instability of the shear layer with the continuous velocity profile, (again in length units scaled on the shear layer thickness and velocity scaled on the maximum velocity).

$$U_o = \frac{1}{2}\left(1 + \tanh(y)\right), \qquad -\infty \le y \le \infty \qquad (12.4.1)$$

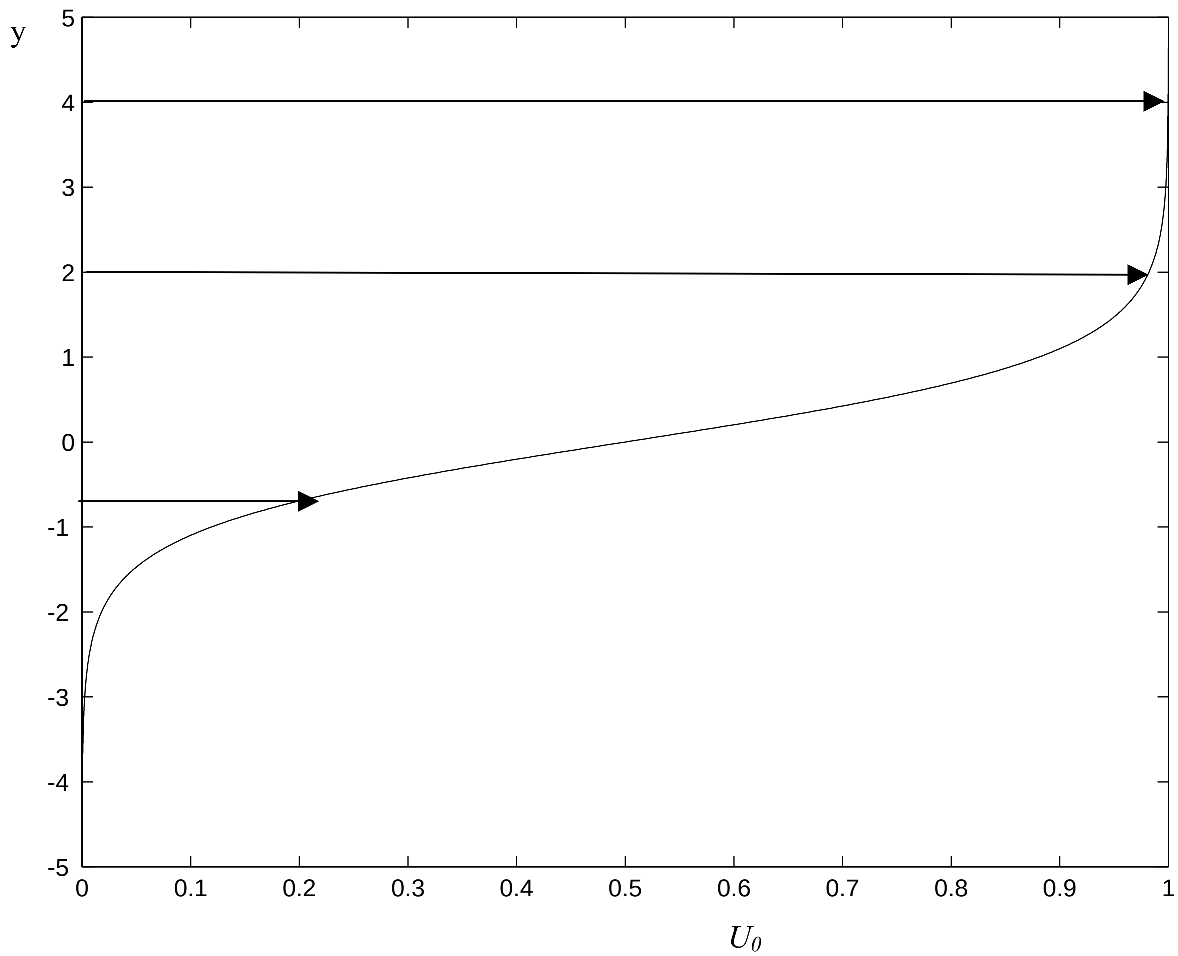

Figure 12.4.1 The tanh velocity profile.

For large,positive y the velocity asymptotes to 1 while for large, negative y the velocity asymptotes to zero. Therefore at large positive y the solution of (12.1.5) (assuming λ =0) is

$$A = const.e^{ily} \tag{12.4.2}$$

where

$$l^2 + k^2 = \frac{\beta}{1-c} \tag{12.4.3}$$

If the solution is a marginally neutral solution the semi-circle theorem tells us that c <1, the maximum basic flow speed. Therefore it is possible for l to be real. On the other hand, for large negative y the solution of the form (12.4.2) would have an l that satisfies,

$$l^2 + k^2 = -\frac{\beta}{c} \tag{12.4.4}$$

so that no real solutions for l can be found. Thus we would anticipate that if we are going to observe radiation from weakly unstable disturbances the radiation will be found for positive y. This is qualitatively what we have already seen in the discontinuous velocity case of the last section.

If there is a disturbance with a small imaginary part to c, or more precisely , if

$$c_i << 1 - c_r \tag{12.4.5}$$

Then if we write,

$$l = l_o + \varepsilon l_1 + ..., \qquad \varepsilon = O(c_i) \tag{12.4.6}$$

then from (12.4.3) at large y,

$$l_o{}^2 + k^2 = \frac{\beta}{1-c_r}, \tag{12.4.7a}$$

and

$$2\varepsilon l_1 l_o = ic_i \frac{\beta}{(1-c_r)^2} = ic_i \frac{\left[k^2 + l_o{}^2\right]^2}{\beta} \tag{12.4.7b}$$

Therefore the form of the solution for large y will be

$$\begin{aligned}\phi = Ae^{-ikct+ikx} &= const.\left[e^{i(kx+l_o y-kc_r t)}\right]\exp(kc_i t - c_i y\frac{(k^2+l_o{}^2)^2}{2\beta l_o}) \\ &= const.\left[e^{i(kx+l_o y-kc_r t)}\right]\exp\left\{-\frac{c_i(k^2+l_o{}^2)^2}{2\beta l_o}\left[y - \frac{2\beta k l_o}{(k^2+l_o{}^2)^2}t\right]\right\}\end{aligned} \tag{12.4.8}$$

So that the disturbance, as $y \rightarrow$ infinity, consists of a plane wave with an envelope of decay whose amplitude propagates outward at the group velocity of that wavenumber. The behavior is shown in the figure below.

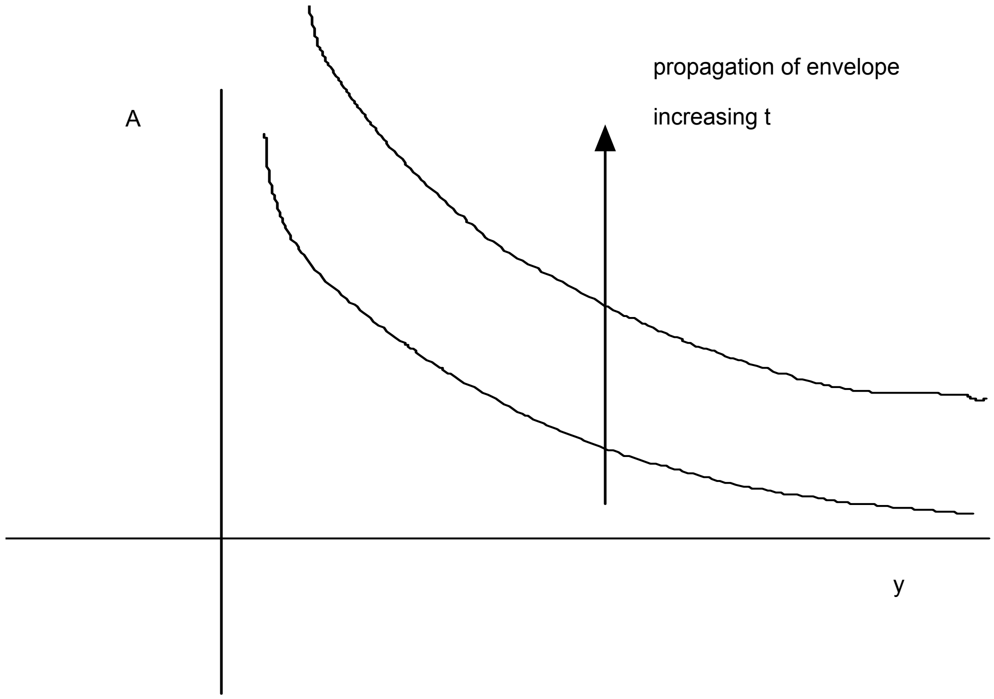

Figure 12.4.2 The amplitude of the disturbance at large y exponentially decreases with y but increases with time.

The reason for the decay in *y* is that the disturbance is radiating outward from its source in the region of the shear where its amplitude is increasing exponentially with time. By the time it reaches large *y* moving with the group velocity the disturbance at smaller *y* has grown larger. The exponential decay is not a sign that the disturbance is trapped, only that the source amplitude is increasing with time so that the amplitude at large *y* is a measure of its amplitude at an earlier time. All of this assumes that a radiating disturbance with a small c_i exists that will radiate to large *y*. Even if such modes have small growth rates they can be very important due to their ability to radiate energy to great distances from the shear zone. The disturbance is self excited and could influence the field of motion far from the source, again, if such disturbances are possible.

The stability of the tanh profile was studied in great detail by both Kuo in the reference already given and by Dickinson and Clare ,(1973. J. Atmos. Science, **30**, 1035-1047). We shall follow the latter reference.

The potential vorticity gradient corresponding to (12.4.1) is

$$q_{o_y} = \beta + \frac{\tanh y}{\cosh y^2} = \beta + \tanh y - \tanh^3 y \qquad (12.4.9)$$

The last two terms in $\tanh y$ have a minimum value of –0.3849 where $\tanh y$=-0.577. (Just find the stationary points of $z-z^3$). Thus instability requires β< 0.3849. For positive β the zeros of the potential vorticity will occur for $y < 0$. Figure 12.4.3 shows a graph of the pv gradient for $\beta = 0.2$, for example.

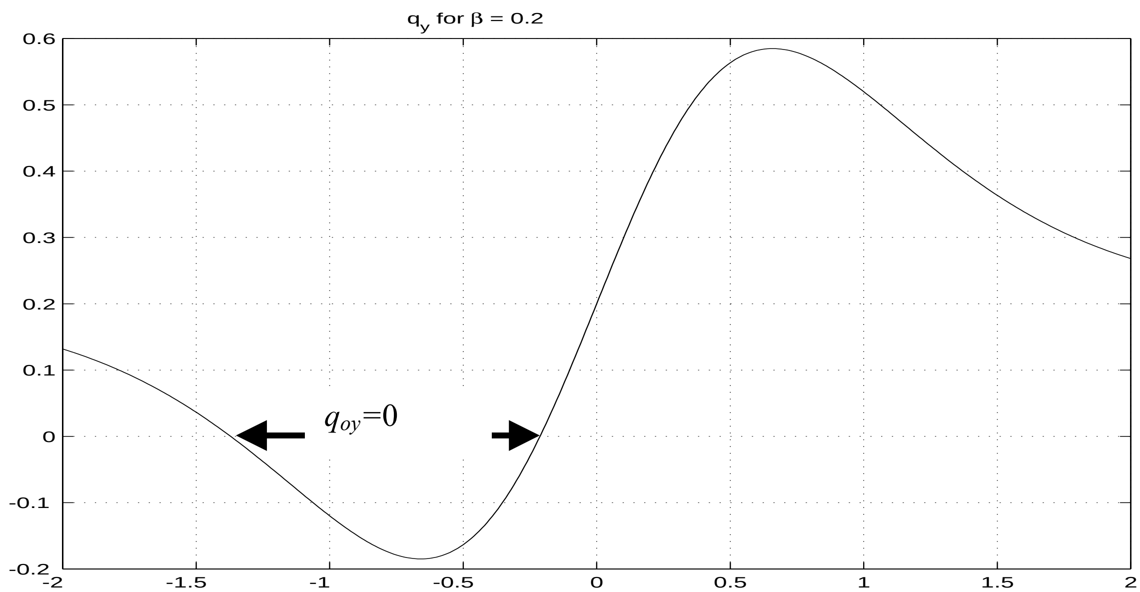

Figure 12.4.3 The potential vorticity gradient for the tanh profile for β =0.2

We can find non singular neutral solutions by the methods of the earlier sections by looking for solutions whose phase speeds correspond to the basic flow velocity at the points where the pv gradient vanishes. Skipping the details, the solution can be found in the form,

$$A = e^{\mu y} / \cosh y \qquad (12.4.10)$$

if,

$$\mu = \left(1 - k^2\right)^{1/2},$$

$$c = 1/2 - \mu/2, \qquad (12.4.11\ a,b,c,d)$$

$$\beta = k^2\mu = k^2\left(1 - k^2\right)^{1/2}$$

$$\rightarrow c = \frac{1}{2} - \frac{\beta}{2k^2}$$

This implies that for positive β the disturbance has its critical layer for $y < 0$. Combining (1.2.4 a) and (12.4.c) we obtain the curve in the β, k plane on which the marginal disturbance exists. That curve is shown in figure 12.4.4

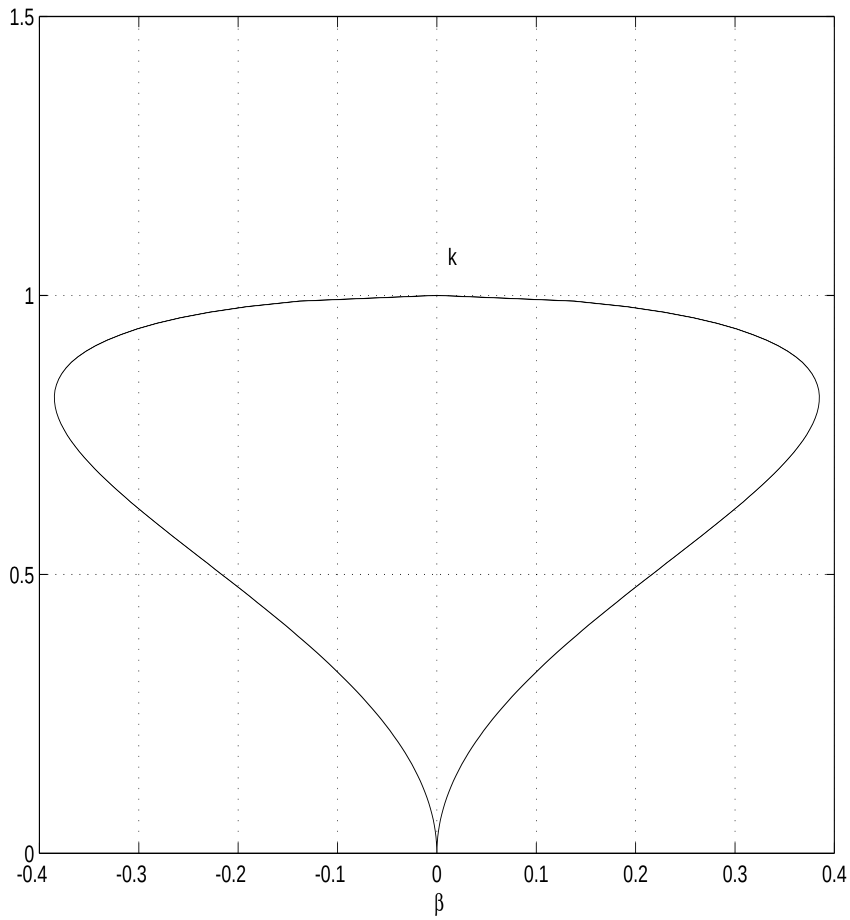

Figure 12.4.4 The curve defining the locus of nonsingular neutral solutions.

From the perturbation method of section 12.1 one can demonstrate that unstable modes exist within the domain enclosed by the curve in figure 12.4.4. The important question is whether in this case that curve is truly the boundary of unstable solutions or whether there are unstable solutions outside that curve which may be radiating and, by necessity be singular. This is the problem discussed both by Dickinson and Clare. In the their paper they present the results of a very careful and detailed numerical study. The numerical problem is delicate since one is looking for solutions with small c_i and a singularity which is therefore just off the real y line. Great care must be taken to resolve the rapid variations of the solution in the vicinity of the critical point. You are referred to their paper for a detailed discussion of their methods. Suffice it to say they used two independent methods as a check. In one method they discretized the problem making it a matrix eigenvalue problem for c. In the other case they used a "shooting" method to solve the ODE iteratively adjusting c until boundary conditions at large positive and negative y were satisfied. Here, their results are presented. The figures come from their paper and they use the notation α in place of k for wavenumber.

Figure 12.4.5 shows the contours of growth rate and phase speed within the boundary traced out by the neutral curve of the nonsingular mode. However, the curve is shown only for wave numbers such that $k^2 > 0.13$. For such wavenumbers the curve given by (12.4.4) is the stability boundary.

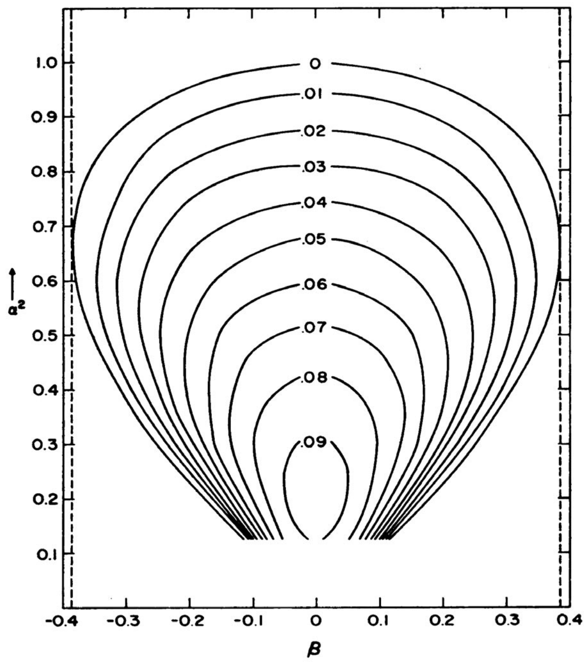

FIG. 1. Contours of growth rate αc_i of the dominant unstable root for $\alpha^2 > 0.13$.